MAC 2311 (Calculus I)
TEST 4, Friday November 20, 2015

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University administration. Total= 60 points. Good Luck!

1. [8] Solve the initial-value problem: $\begin{cases} \dfrac{dy}{dx} = xe^{x^2} - 2x^3 \\ y(0) = 3. \end{cases}$

First, get the general solution which depends on a constant, then use the initial condition $y(0)=3$ to get the constant.

$y = \int (xe^{x^2} - 2x^3)dx = \int xe^{x^2}dx - \int 2x^3 dx$

$u = x^2;\ du = 2x\,dx$

$= \int e^u \frac{du}{2} - 2\frac{x^4}{4} = \frac{e^u}{2} - \frac{x^4}{2} + C = \frac{e^{x^2}}{2} - \frac{x^4}{2} + C$

$y(x) = \frac{e^{x^2}}{2} - \frac{x^4}{2} + C.$

$y(0) = \frac{e^0}{2} - 0 + C = \frac{1}{2} + C = 3 \longrightarrow C = 3 - \frac{1}{2} = \frac{5}{2}$; hence

$y = \frac{e^{x^2}}{2} - \frac{x^4}{2} + \frac{5}{2}$

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2. [24] Evaluate the following indefinite integrals.

a) $\int \sin x(5\csc^3 x - 6\cot x)\, dx =$

$= \int (5\csc^2 x - 6\cos x)dx$

$= -5\cot x - 6\sin x + C$

b) $\int (7e^x - 4\cos x)^{1120}(7e^x + 4\sin x)\, dx =$

$u = 7e^x - 4\cos x,\ du = (7e^x + 4\sin x)dx$

$= \int u^{1120} du$

$= \frac{u^{1121}}{1121} + C = \frac{(7e^x - 4\cos x)^{1121}}{1121} + C$

c) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx =$

$u = e^x,\ du = e^x dx$

$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C$

$= \sin^{-1}(e^x) + C$

d) $\int \frac{\sqrt{t}}{1+t}\, dt =$

$u = \sqrt{t} \to t = u^2 \to dt = 2u\,du$

$\int \frac{u(2u)du}{1+u^2} = 2\int \frac{u^2+1-1}{1+u^2} du$

$= 2\int (1 - \frac{1}{1+u^2}) du$

$= 2(u - \tan^{-1} u) + C$

$= 2(\sqrt{t} - \tan^{-1}(\sqrt{t})) + C$

e) $\int (\frac{3}{x\sqrt{x^2-1}} - 5^{-x})\, dx =$

$= 3\sec^{-1}(|x|) - \int (\frac{1}{5})^x dx$

$= 3\sec^{-1}(|x|) - \frac{(\frac{1}{5})^x}{\ln(1/5)} + C$

$= 3\sec^{-1}(|x|) + \frac{5^{-x}}{\ln 5} + C$

f) $\int \frac{5x^4 - 3x^3 + x^2 - 2x}{x^2}\, dx = \int (5x^2 - 3x + 1 - \frac{2}{x}) dx$

$= \frac{5}{3}x^3 - \frac{3}{2}x^2 + x - 2\ln|x| + C$

3. [10] a) State the Mean Value Theorem. b) Show that the function f defined by $f(x) = 2x - \arcsin(x)$, x in $[0, 1]$, satisfies all the requirements of the Mean Value Theorem. c) Find all numbers x_0 in $(0, 1)$ such that $f'(x_0) = \frac{f(1)-f(0)}{1-0}$.

a) See text or notes

b) f is the difference of two functions that are both continuous on $[0,1]$ and differentiable on $(0,1)$; so does f. Hence f satisfies all the requirements of the MVT.

c) $f'(x) = 2 - \frac{1}{\sqrt{1-x^2}}$ $\quad f'(x_0) = 2 - \frac{1}{\sqrt{1-x_0^2}} = \frac{f(1)-f(0)}{1-0} = 2 - \frac{\pi}{2}$

So $-\frac{1}{\sqrt{1-x_0^2}} = -\frac{\pi}{2}$ or $\sqrt{1-x_0^2} = \frac{2}{\pi}$; so $1 - x_0^2 = \frac{4}{\pi^2}$ or

$x_0^2 = 1 - \frac{4}{\pi^2}$.

Hence $\underline{x_0 = \sqrt{1-\frac{4}{\pi^2}}}$ or $x_0 = \cancel{-\sqrt{1-\frac{4}{\pi^2}}}$ not in $(0,1)$

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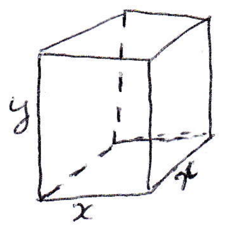
4. [10, **Bonus**] A closed rectangular container with a square base is to have a volume of 1440 ft^3. Find the dimensions of the container with minimum surface area.



$V = \text{volume} = x^2 y = 1440 \to y = \frac{1440}{x^2}$

$A = \text{Surface area} = 2x^2 + 4xy$

$A(x) = 2x^2 + 4x\frac{(1440)}{x^2} = 2x^2 + \frac{4(1440)}{x}$, $x > 0$

$A'(x) = 4x - \frac{4(1440)}{x^2} = \frac{4(x^3 - 1440)}{x^2}$

$A'(x) = 0 \to x = \sqrt[3]{1440}$; $A''(x) = 4 + \frac{8(1440)}{x^3} > 0$

So A has a single local minimum at $x = \sqrt[3]{1440}$ by SDT; so A is the least for $x = \sqrt[3]{1440}$ and $y = \sqrt[3]{1440}$.

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5. [8] The function $s(t) = t^3 - 6t^2 + 9t + 1$, $t \geq 0$ denotes the position of a particle moving along a straight line, where s is in feet and t in seconds. a) find the velocity and acceleration functions. b) When is the particle stopped? c) When is the particle speeding up? Slowing down? d) Find the total distance traveled from time $t = 0$ to time $t = 2$. e) Give a schematic picture of the motion.

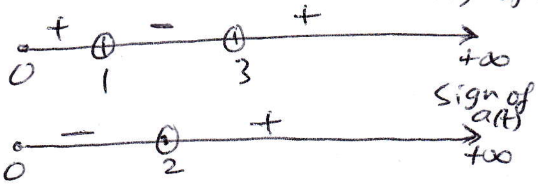
a) $v(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-1)(t-3)$

$a(t) = 6t - 12 = 6(t-2)$

b) Particle is stopped when $v(t) = 0$; $3(t-1)(t-3) = 0$; So $t = 1$ or $t = 3$

Particle is stopped for $t = 1$ and $t = 3$.

c)



- Particle is
- Speeding up on $(1,2) \cup (3,+\infty)$
- Slowing down on $(0,1) \cup (2,3)$

d) $d = |s(0) - s(1)| + |s(1) - s(2)| = |1-5| + |5-3| = 6\text{ ft}$

e)