MAC 2311 (Calculus I) — Key
TEST 4, Friday April 17, 2015

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University administration. Total= 62 points. Good Luck!

1. [10] Consider the parametric curve given by $x = t^4 - t^2 + 2$, $y = t^6 - t^4 + 5$.
a) Find $\frac{dy}{dx}$. b) Find the equation of the tangent line to the curve at the point obtained when $t = 1$. c) If a particle is moving on that curve, for what value(s) of the parameter is it moving horizontally? Vertically?

a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^5-4t^3}{4t^3-2t}$; b) $\left.\frac{dy}{dx}\right|_{t=1} = \frac{6-4}{4-2} = \frac{2}{2} = 1 =$ slope of tangent line

When $t=1$, $x=2$, and $y=5$; Equation of tangent line: $y = 5 + (x-2)$

c) particle is moving horizontally when $\frac{dy}{dt} = 0$, and $\frac{dx}{dt} \neq 0$

Now $\frac{dy}{dt} = 0$ for $t^3(6t^2-4) = 0$ or $t=0$ and $t = \pm\sqrt{\frac{4}{6}} = \pm\sqrt{\frac{2}{3}}$

$\frac{dx}{dt} = 0$ for $2t(2t^2-1) = 0$ or $t=0$ and $t = \pm\sqrt{\frac{1}{2}}$

Hence particle is moving horizontally when $t = \pm\sqrt{2/3}$, and particle is moving vertically when $t = \pm\sqrt{1/2}$.

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2. [24] Evaluate the following indefinite integrals.

a) $\int (\frac{5}{\sin^2 x} - 4\cos x)\,dx = \int (5\csc^2 x - 4\cos x)\,dx$
$= -5\cot x - 4\sin x + C$
$= -5\cot x - 4\sin x + C$

b) $\int (x^3 + 2\tan x)^{2015}(3x^2 + 2\sec^2 x)\,dx = \int u^{2015}\,du$
$u = x^3 + 2\tan x$
$du = 3x^2 + 2\sec^2 x\,dx$
$= \frac{u^{2016}}{2016} + C$
$= \frac{(x^3+2\tan x)^{2016}}{2016} + C$

c) $\int x^{\frac{3}{4}}(x+2)^2\,dx = \int x^{\frac{3}{4}}(x^2+4x+4)\,dx$
$= \int (x^{\frac{11}{4}} + 4x^{\frac{7}{4}} + 4x^{\frac{3}{4}})\,dx$
$= \frac{4}{15}x^{\frac{15}{4}} + \frac{16}{11}x^{\frac{11}{4}} + \frac{16}{7}x^{\frac{7}{4}} + C$

d) $\int \frac{x^4+x^2-2}{x^2+1}\,dx = \int (\frac{x^2(x^2+1)}{x^2+1} - \frac{2}{x^2+1})\,dx$
$= \int (x^2 - \frac{2}{x^2+1})\,dx$
$= \frac{x^3}{3} - 2\tan^{-1}x + C$

e) $\int (3^x + 2e^x - \frac{4}{\sqrt{1-x^2}})\,dx = \frac{3^x}{\ln 3} + 2e^x - 4\sin^{-1}x + C$
$= \frac{3^x}{\ln 3} + 2e^x + 4\cos^{-1}x + C$

f) $\int x\sqrt{x-2}\,dx = \int (u+2)\sqrt{u}\,du$
$u = x-2$
$du = dx$
$= \int (u^{\frac{3}{2}} + 2u^{1/2})\,du$
$= \frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} + C$
$= \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$

3. [10] a) State the Mean Value Theorem. b) Show that the function f defined by $f(x) = \frac{x}{x+2}$, x in $[1, 2]$, satisfies all the requirements of the Mean Value Theorem. c) Find all numbers x_0 in $(1, 2)$ such that $f'(x_0) = \frac{f(2)-f(1)}{2-1}$.

a) See Notes or text.

b) f is a rational function whose denominator does not vanish on $[1,2]$; so f is continuous on $[1,2]$ and differentiable on $(1,2)$. Hence f satisfies all the hypotheses of the MVT.

c) $f'(x) = \frac{1(x+2)-1(x)}{(x+2)^2} = \frac{2}{(x+2)^2}$. $f(2) = \frac{2}{4} = \frac{1}{2}$, $f(1) = \frac{1}{3}$

$\frac{2}{(x_0+2)^2} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$; so $2(6) = (x_0+2)^2$

$12 = (x_0+2)^2$. So $x_0+2 = \pm\sqrt{12} = \pm 2\sqrt{3}$

$x_0 = 2\sqrt{3} - 2$ lies in $(1,2)$, but

$x_0 = -2\sqrt{3} - 2$ does not

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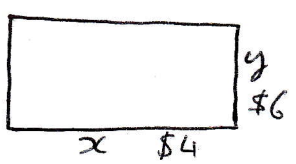
4. [10] Find the maximum rectangular area that can be fenced with \$3600 if two opposite sides of the rectangle will use fencing costing \$4 per foot and the remaining sides will use fencing costing \$6 per foot.



A = area $= xy$

C = Cost $= 2x(4) + 2y(6) = 8x + 12y = 3600$

So $2x + 3y = 900$; solving for y: $y = 300 - \frac{2}{3}x$

$A(x) = x\left(300 - \frac{2}{3}x\right)$, $0 \le x \le 450$, as $300(3) - 2x \ge 0$

$A'(x) = 300 - \frac{4}{3}x$; $A'(x) = 0 \rightarrow 4x = 900 \rightarrow x = 225$

$A''(x) = -\frac{4}{3} < 0$; $A(225)$ is a maximum. $y = 300 - \frac{450}{3} = 150$

Maximum area $= 225(150)\ ft^2$

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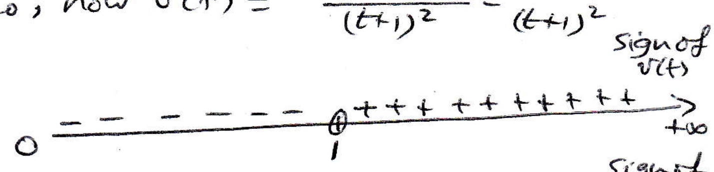
5. [8] The function $s(t) = \frac{2}{t+1} + \ln(1+t)$, $t \ge 0$ denotes the position of a particle moving along a straight line, where s is in feet and t in seconds. a) find the velocity and acceleration functions. b) When is the particle stopped? c) When is the particle speeding up? Slowing down? d) Find the total distance traveled from time $t = 0$ to time $t = 3$. e) Give a schematic picture of the motion.

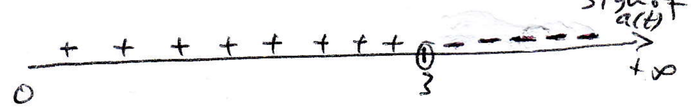
a) $v(t) = s'(t) = -\frac{2}{(t+1)^2} + \frac{1}{t+1}$ $\quad a(t) = v'(t) = \frac{4}{(t+1)^3} - \frac{1}{(t+1)^2}$

b) Particle is stopped when $v(t) = 0$; now $v(t) = \frac{-2+(t+1)}{(t+1)^2} = \frac{t-1}{(t+1)^2}$

$v(t) = 0$ for $t = 1$.



c) $a(t) = \frac{4-(t+1)}{(t+1)^3} = \frac{3-t}{(t+1)^3}$



• particle is speeding up when $v(t)$ and $a(t)$ have the same sign; on $(1,3)$

• particle is slowing down when $v(t)$ and $a(t)$ have opposite signs; on $(0,1) \cup (3,+\infty)$

d) Total distance $= |s(1)-s(0)| + |s(3)-s(1)| = |1+\ln 2 - 2| + \left|\frac{1}{2} + \ln 4 - (1+\ln 2)\right|$

$= 1 - \ln 2 - \frac{1}{2} + \ln 2 = \frac{1}{2}$; as $\ln 4 = 2\ln 2$

e)