MAC 2313 (Calculus III)
Test 4, Friday November 20, 2015 — Answers

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question; answers without any explanation won't get any credits. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Total =65 points. Good luck.

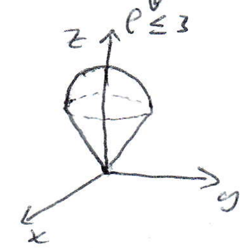
1. [12] Use spherical coordinates to find the volume and centroid of the solid bounded below by the cone $z = \sqrt{3x^2+3y^2}$, and above by the sphere $x^2+y^2+z^2=9$.

$x^2+y^2=\rho^2\sin^2\phi$, $\sqrt{3(x^2+y^2)} \le z \le \sqrt{9-(x^2+y^2)} \to \sqrt{3}\rho\sin\phi \le \rho\cos\phi \le \sqrt{9-\rho^2\sin^2\phi}$

$\tan\phi \le \frac{1}{\sqrt{3}}$, $\phi \le \frac{\pi}{6}$ | $\rho^2\cos^2\phi \le 9-\rho^2\sin^2\phi$, $\rho^2 \le 9$, $\rho \le 3$

$$V=\int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = 2\pi\left[-\cos\phi\right]_0^{\pi/6}\left[\frac{\rho^3}{3}\right]_0^3$$
$$= 2\pi\left(-\frac{\sqrt{3}}{2}+1\right)(9) = 9\pi(2-\sqrt{3})$$

Solid has z-axis as axis of symmetry; so, if centroid is $(\bar{x},\bar{y},\bar{z})$, then $\bar{x}=0=\bar{y}$.



$$\bar{z}=\frac{1}{V}\int_0^{2\pi}\int_0^{\pi/6}\int_0^3 \rho^2\sin\phi\,\rho\cos\phi\, d\rho\, d\phi\, d\theta$$
$$=\frac{2\pi}{V}\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/6}\left[\frac{\rho^4}{4}\right]_0^3 = \frac{1}{9(2-\sqrt{3})}\left(\frac{1}{4}\right)\left(\frac{81}{4}\right)=\frac{9}{16(2-\sqrt{3})}$$

$$\left(0,0,\frac{9(2+\sqrt{3})}{16}\right)$$

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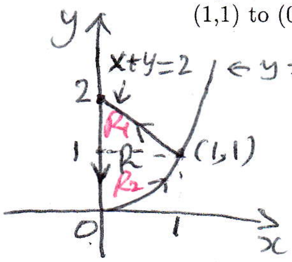
2. [10] a) State Green's Theorem.

See text or notes

b) Use it to evaluate the line integral $\int_C(-5y+\ln(1+e^{(x^5)}))\,dx+(4x-3^{(y^4)})\,dy$, where C is composed of the segment of line joining (0,2) to (0,0), the arc of the parabola $y=x^2$ from (0,0) to (1,1), and the line segment from (1,1) to (0,2).



$$I=\int_C(-5y+\ln(e^{x^5}))dx+(4x-3^{(y^4)})dy$$
$$=\iint_R(\partial_x(4x-3^{(y^4)})-\partial_y(-5y+\ln(e^{x^5})))dA$$
$$=\iint_R(4+5)\,dA = 9\iint_R dA = 9\,(\text{area of } R)$$

Area of $R_1 = \frac{1(1)}{2}=\frac{1}{2}$, R_1 = triangular region; base = 1 = height

Area of $R_2 = \int_0^1(1-x^2)dx = x-\frac{x^3}{3}\Big]_0^1 = 1-\frac{1}{3}=\frac{2}{3}$

Hence $I = 9\left(\frac{1}{2}+\frac{2}{3}\right)=\frac{21}{2}$, or $I=\int_0^1\int_{x^2}^{2-x} 9\,dy\,dx$
$$=9\int_0^1(2-x-x^2)dx$$
$$=9\left(2x-\frac{x^2}{2}-\frac{x^3}{3}\right)\Big]_0^1=\frac{21}{2}$$

3. [10] a) Find the flux Φ of the vector $\mathbf{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$ across σ if σ is the parametric surface given by $\vec{r}(u,v) = u^2\vec{i} + u\cos v\,\vec{j} + u\sin v\,\vec{k}$, for $0 \le u \le 2$ and $0 \le v \le \pi$.

$$\Phi = \iint_\sigma \vec F\cdot\vec n\,dS, \qquad \vec n = \vec r_u \times \vec r_v = \begin{vmatrix} \vec i & \vec j & \vec k \\ 2u & \cos v & \sin v \\ 0 & -u\sin v & u\cos v \end{vmatrix}$$

$$= u(\cos^2 v + \sin^2 v)\vec i - (2u^2\cos v - 0)\vec j + (-2u^2\sin v - 0)\vec k$$

$$= u\vec i - 2u^2\cos v\,\vec j - 2u^2\sin v\,\vec k$$

$$= \int_0^\pi\int_0^2 (u^2\vec i + u\cos v\,\vec j + u\sin v\,\vec k)\cdot(u\vec i - 2u^2\cos v\,\vec j - 2u^2\sin v\,\vec k)\,du\,dv$$

$$= \int_0^\pi\int_0^2 u^3 - 2u^3(\cos^2 v + \sin^2 v)\,du\,dv$$

$$= \int_0^\pi\int_0^2 (u^3 - 2u^3)\,du\,dv = \pi\left[-\frac{u^4}{4}\right]_0^2 = -4\pi$$

b) Describe the portion of the paraboloid $z = 6 - x^2 - y^2$ on or above the plane $z = 2$ in terms of the parameters r and θ where (r, θ, z) are cylindrical coordinates of a point on the surface.

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 6 - r^2$

$2 \le z \le 6 - r^2 \rightarrow 2 \le 6 - r^2 \rightarrow r^2 \le 6 - 2 = 4 \rightarrow r \le 2$

$0 \le r \le 2, \quad 0 \le \theta \le 2\pi.$

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4. [8] Find the mass M of the surface σ which is the portion of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 3$ if its density is $\delta(x,y,z) = y^2$.

$z_x = \frac{x}{z}, \quad z_y = \frac{y}{z} \qquad z_x^2 + z_y^2 + 1 = \frac{x^2+y^2}{z^2} + 1 = \frac{x^2+y^2}{x^2+y^2} + 1 = 1 + 1 = 2$

$$M = \iint_\sigma \delta(x,y,z)\,dS = \int_0^{2\pi}\sqrt{2}\int_1^3 r^2\sin^2\theta\; r\,dr\,d\theta$$

$dS = \sqrt{z_x^2 + z_y^2 + 1}\,dA$

$= \sqrt{2}\,r\,dr\,d\theta$

$1 \le z \le 3$

$\downarrow$

$1 \le \sqrt{x^2+y^2} \le 3$

$1 \le x^2+y^2 \le 9$

$1 \le r^2 \le 9$

$1 \le r \le 3$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi}\sqrt{2}\left(\frac{1-\cos(2\theta)}{2}\right)d\theta \cdot \left[\frac{r^4}{4}\right]_1^3$$

$$= \sqrt{2}\left(\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right)\Big]_0^{2\pi}\left[\frac{81-1}{4}\right]$$

$$= \sqrt{2}\pi\,(20)$$

$$= 20\pi\sqrt{2}$$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

5. [10] a) Let $\mathbf{F}(x,y,z) = (2x^2 - yz)\vec{\imath} + (3y^2 - 2xz)\vec{\jmath} + (4z^2 - 3xy)\vec{k}$. Find div**F** and curl**F**.

$$\operatorname{div} F = \partial_x(2x^2 - yz) + \partial_y(3y^2 - 2xz) + \partial_z(4z^2 - 3xy)$$
$$= 4x + 6y + 8z$$

$$\operatorname{Curl} F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2x^2 - yz & 3y^2 - 2xz & 4z^2 - 3xy \end{vmatrix} = (-3x - (-2x))\vec{i} - (-3y - (-y))\vec{j} + (-2z - (-z))\vec{k}$$
$$= -x\vec{i} + 2y\vec{j} - z\vec{k}$$

b) Find the length L of the arc of the curve C parametrized by $\vec{r}(t) = \cos^3 t\,\vec{\imath} + \sin^3 t\,\vec{\jmath}$, $0 \le t \le \pi/2$.

$$L = \int_C ds = \int_0^{\pi/2} \|r'(t)\| dt = \int_0^{\pi/2} \sqrt{(-3\sin t\cos^2 t)^2 + (3\cos t\sin^2 t)^2}\, dt$$
$$= \int_0^{\pi/2} \sqrt{9\sin^2 t\cos^4 t + 9\cos^2 t\sin^4 t}\, dt$$
$$= 3\int_0^{\pi/2} \sqrt{\sin^2 t\cos^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}}\, dt$$
$$= 3\int_0^{\pi/2} \sin t\cos t\, dt$$
$$= 3\left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{3}{2}$$

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6. [15] Let $\mathbf{F}(x,y) = (e^x \cos y + x^4)\vec{\imath} - (e^x \sin y - y^5)\vec{\jmath}$. a) Show that **F** is conservative. b) Find a potential function φ for **F**. c) Evaluate the line integral $\int_C (e^x \cos y + x^4)\,dx - (e^x \sin y - y^5)\,dy$ along the curve C parametrized by $\vec{r}(t) = \sqrt{1+t}\,\vec{\imath} + \sin^{-1} t\,\vec{\jmath}$, $0 \le t \le 1$.

a) $\partial_y(e^x\cos y + x^4) = -e^x\sin y = \partial_x(-e^x\sin y + y^5)$; so F is conservative

b) $F = \nabla\varphi$; so (i) $\varphi_x(x,y) = e^x\cos y + x^4$, (ii) $\varphi_y(x,y) = -e^x\sin y + y^5$

integrating (i) w.r.t. x:

$\varphi(x,y) = \int (e^x\cos y + x^4)dx = e^x\cos y + \frac{x^5}{5} + k(y)$ (iii)

Differentiate (iii) w.r.t. y:

$\varphi_y(x,y) = -e^x\sin y + k'(y)$
$= -e^x\sin y + y^5$, by (ii)

So $k'(y) = y^5$; hence $k(y) = \int y^5 dy = \frac{y^6}{6} + C$, C = constant

We may choose $C = 0$; So $\varphi(x,y) = e^x\cos y + \frac{x^5}{5} + \frac{y^6}{6}$

c) $\vec{r}(1) = (\sqrt{2}, \frac{\pi}{2})$, $\vec{r}(0) = (1,0)$

$$\int_C F \cdot d\vec{r} \underset{\text{by FTLI}}{=} \varphi(\sqrt{2}, \tfrac{\pi}{2}) - \varphi(1,0) = e^{\sqrt{2}}\cos\frac{\pi}{2} + \frac{4\sqrt{2}}{5} + \frac{\pi^6}{64(6)} - \left(e\cos 0 + \frac{1}{5}\right)$$
$$= \frac{4\sqrt{2} - 1}{5} + \frac{\pi^6}{64(6)} - e$$