MAC 2313 (Calculus III) — Answers
Test 4, Friday June 09, 2017

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work. Guessing the correct answers won't give you any credits. You must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may use the back of the page to show your work. 2 pages. Total=60 points. Always do your best.

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1. [20] a) Let $\mathbf{F}(x,y,z) = (2x^2 - yz)\vec{i} + (3y^2 - 2xz)\vec{j} + (4z^2 - 3xy)\vec{k}$. Find div$\mathbf{F}$ and curl$\mathbf{F}$.

$$\text{div}F = \partial_x(2x^2 - yz) + \partial_y(3y^2 - 2xz) + \partial_z(4z^2 - 3xy)$$
$$= 4x + 6y + 8z$$

$$\text{Curl}F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2x^2 - yz & 3y^2 - 2xz & 4z^2 - 3xy \end{vmatrix} = (\partial_y(4z^2 - 3xy) - \partial_z(3y^2 - 2xz))\vec{i} - (\partial_x(4z^2 - 3xy) - \partial_z(2x^2 - yz))\vec{j} + (\partial_x(3y^2 - 2xz) - \partial_y(2x^2 - yz))\vec{k}$$

$$= -x\vec{i} + 2y\vec{j} - z\vec{k} \qquad = (-3x + 2x)\vec{i} - (-3y + y)\vec{j} + (-2z + z)\vec{k}$$

b) Find parametric equations in terms of the parameters θ and ϕ, where (ρ, θ, ϕ) are spherical coordinates of a point on the portion of the sphere $x^2 + y^2 + z^2 = 4$ on or above the plane $z = 1$.

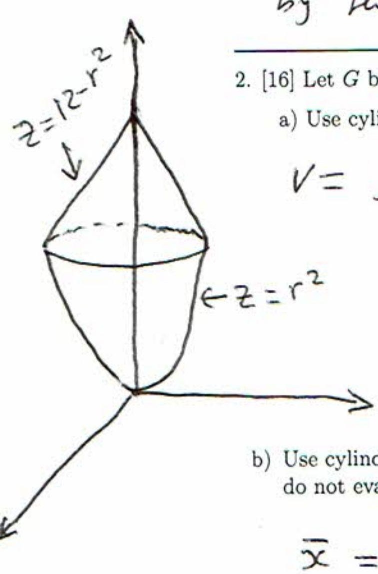
$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$. Now $x^2+y^2+z^2 = \rho^2$; so $\rho^2 = 4$ or $\rho = 2$

Hence $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$

$1 \le z \le \sqrt{4 - x^2 - y^2} \longrightarrow 1 \le 2\cos\phi \le \sqrt{4 - 4\sin^2\phi} = 2\sqrt{1 - \sin^2\phi} = 2\sqrt{\cos^2\phi} = 2\cos\phi$

$\hookrightarrow \frac{1}{2} \le \cos\phi \longrightarrow 0 \le \phi \le \frac{\pi}{3}$

The projection of the surface on the xy-plane is the region enclosed by the circle $x^2+y^2 = 3$; so $0 \le \theta \le 2\pi$.



2. [16] Let G be the solid bounded above by the cone $z = 12 - \sqrt{x^2 + y^2}$ and below by the paraboloid $z = x^2 + y^2$.

a) Use cylindrical coordinates to evaluate the volume of the solid G. At intersection, $12 - r = r^2$ or
$r^2 + r - 12 = 0$
$(r+4)(r-3) = 0$
so $r = 3$, as $r \ge 0$

$$V = \iiint_G dV = \int_0^{2\pi}\int_0^3\int_{r^2}^{12-r} r\,dz\,dr\,d\theta$$
$$= 2\pi\int_0^3 r(12 - r - r^2)\,dr$$
$$= 2\pi\left[6r^2 - \frac{r^3}{3} - \frac{r^4}{4}\right]_0^3 = 2\pi\left(54 - 9 - \frac{81}{4}\right)$$

b) Use cylindrical coordinates to set up a triple integral for each of the coordinates of the centroid of the solid G, but do not evaluate any of those integrals

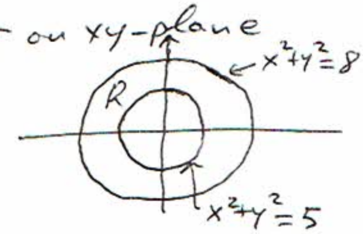
$$\bar{x} = \frac{1}{V}\int_0^{2\pi}\int_0^3\int_{r^2}^{12-r} r^2\cos\theta\,dz\,dr\,d\theta$$

$$\bar{y} = \frac{1}{V}\int_0^{2\pi}\int_0^3\int_{r^2}^{12-r} r^2\sin\theta\,dz\,dr\,d\theta$$

$$\bar{z} = \frac{1}{V}\int_0^{2\pi}\int_0^3\int_{r^2}^{12-r} zr\,dz\,dr\,d\theta$$

3. [10] Evaluate the surface integral $\iint_\sigma \sqrt{x^2+y^2+z^2}\,dS$ if σ is the portion of the sphere $x^2+y^2+z^2=9$ between the planes $z=1$ and $z=2$.

$$\partial_x(x^2+y^2+z^2) = \partial_x(9)=0, \quad 2x+2z_x z = 0 \to z_x = -\frac{x}{z}$$
$$\partial_y(x^2+y^2+z^2) = \partial_y(9)=0, \quad 2y+2z_y z = 0 \to z_y = -\frac{y}{z}$$
$$z_x^2+z_y^2+1 = \frac{x^2+y^2+z^2}{z^2} = \frac{9}{9-x^2-y^2}$$
$$\iint_\sigma \sqrt{x^2+y^2+z^2}\,dS = \iint_R \sqrt{9}\sqrt{\frac{9}{9-x^2-y^2}}\,dA, \quad R = \text{projection of } \sigma \text{ on } xy\text{-plane}$$
$$= 9\int_0^{2\pi}\int_{\sqrt5}^{\sqrt8} \frac{r}{\sqrt{9-r^2}}\,dr\,d\theta$$
$$= 18\pi\left[-\sqrt{9-r^2}\right]_{\sqrt5}^{\sqrt8} = 18\pi(-1+2) = 18\pi$$



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4. [14] Use the change of variables $u=x-y$ and $v=x+y$ to evaluate the integral $\iint_{\mathcal{R}} \frac{(x-y)^3}{1+(x+y)^2}\,dA$, where $\mathcal{R}$ is the rectangular region enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$ and $x+y=1$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1=2\ ; \text{ So } \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}. \quad \text{Note } 0\le u\le 2,\ 0\le v\le 1$$

Hence

$$\iint_R \frac{(x-y)^3}{1+(x+y)^2}\,dA = \frac{1}{2}\int_0^2\int_0^1 \frac{u^3}{1+v^2}\,dv\,du = \frac{1}{2}\int_0^2 u^3\,du \int_0^1 \frac{dv}{1+v^2}$$
$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_0^2 \left[\tan^{-1}(v)\right]_0^1$$
$$= \frac{1}{2}\left(\frac{2^4}{4}\right)\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$
$$= 2\left(\frac{\pi}{4}\right), \text{ as } \tan^{-1}(0)=0$$
$$= \pi/2$$

Method 2 for Problem 3. Note σ = portion of sphere $x^2+y^2+z^2=9$, so $\rho=3$

$$\vec r(\theta,\phi) = 3\sin\phi\cos\theta\,\vec i + 3\sin\phi\sin\theta\,\vec j + 3\cos\phi\,\vec k$$
$$1 \le 3\cos\phi \le 2 \to \frac13 \le \cos\phi \le \frac23 \to \cos^{-1}(2/3) \le \phi \le \cos^{-1}(1/3);\ 0\le\theta\le 2\pi$$
$$\vec r_\theta \times \vec r_\phi = 9\begin{vmatrix} \vec i & \vec j & \vec k \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix} = 9\Big[(-\sin^2\phi\cos\theta)\vec i - (\sin^2\phi\sin\theta)\vec j - \sin\phi\cos\phi\,\underbrace{(\sin^2\theta+\cos^2\theta)}_{=1}\vec k\Big]$$
$$\|\vec r_\theta \times \vec r_\phi\| = 9\sqrt{\sin^4\phi\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1} + \sin^2\phi\cos^2\phi} = 9\sin\phi\sqrt{\underbrace{\sin^2\phi+\cos^2\phi}_{=1}} = 9\sin\phi$$
$$\iint_\sigma \sqrt{x^2+y^2+z^2}\,dS = \int_0^{2\pi}\int_{\cos^{-1}(2/3)}^{\cos^{-1}(1/3)} 3(9\sin\phi)\,d\phi\,d\theta = 54\pi\left[-\cos\phi\right]_{\cos^{-1}(2/3)}^{\cos^{-1}(1/3)}$$
$$= 54\pi\left(-\cos(\cos^{-1}(1/3)) + \cos(\cos^{-1}(2/3))\right)$$
$$= 54\pi\left(-\frac13+\frac23\right) = \frac{54\pi}{3} = 18\pi.$$