On the controllability of some systems of wave equations.

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Overview

- Wave equations with internal coupling.

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- Some open problems.

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- Wave equations with boundary coupling.

Notations

Ω= bounded domain in $\mathbb{R}^N$, $N \geq 1$,

Γ= boundary of Ω is smooth,

$T > 0$, $Q = \Omega \times (0, T)$

$\omega =$ nonvoid open subset in Ω.

a, b, c, d lie in $L^\infty(0, T; L^s(\Omega))$, $s \geq \max(2, N)$ for $N \neq 2$, and $s > 2$ for $N = 2$.

Controllability

Consider the controllability problems: Given (z^0, z^1) and (w^0, w^1), and $\varepsilon > 0$, find a control h such that if (z, w) solves the system

$$\begin{cases} z_{tt} - \Delta z + az + cw = h1_\omega \text{ in } Q \\ w_{tt} - \Delta w + bz + dw = 0 \text{ in } Q \\ z = 0, \quad w = 0 \text{ on } \Sigma = \partial\Omega \times (0, T) \\ z(0) = z^0; \quad z_t(0) = z^1 \quad w(0) = w^0; \quad w_t(0) = w^1 \text{ in } \Omega, \end{cases}$$

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then (exact controllability)

$$z(T) = 0, \quad z_t(T) = 0, \quad w(T) = 0, \quad w_t(T) = 0 \text{ in } \Omega,$$

or else (approximate controllability)

$$||z(T)||_1 + ||z_t(T)||_2 \leq \varepsilon, \quad ||w(T)||_1 + ||w_t(T)||_2 \leq \varepsilon.$$

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- For approximate controllability, only T must be large enough.
- Lions' HUM reduces exact controllability to an inverse (observability) estimate for the adjoint system.

Observability estimates

Consider the coupled (adjoint) system

$$\begin{cases} u_{tt} - \Delta u + au + bv = 0 \text{ in } Q \\ v_{tt} - \Delta v + cu + dv = 0 \text{ in } Q \\ u = 0, \quad v = 0 \text{ on } \Sigma = \partial\Omega \times (0, T) \\ u(0) = u^0; \quad u_t(0) = u^1 \quad v(0) = v^0; \quad v_t(0) = v^1 \text{ in } \Omega. \end{cases}$$

The coupled system is well-posed in $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$.

Introduce the energies:

$$E_u(t) = \frac{1}{2}\int_\Omega \{|u_t(x,t)|^2 + |\nabla u(x,t)|^2\}\,dx,$$

$$\widehat{E}_u(t) = \frac{1}{2}\left(||u(.,t)||^2_{L^2(\Omega)} + ||u_t(.,t)||^2_{H^{-1}(\Omega)}\right).$$

For each $t \in [0, T]$, set

$$E(t) = E_u(t) + \widehat{E}_v(t), \quad \widehat{E}(t) = \widehat{E}_u(t) + \widehat{E}_v(t).$$

Introduce the function $m(x) = x - x^0$, where x^0 is an arbitrary point in $\mathbb{R}^N$. Set

$$R_1 = \max\left\{|m(x)|; x \in \bar{\Omega}\right\}.$$

Let ν be the unit normal pointing into the exterior of Ω, and set

$$\Gamma_0 = \{x \in \partial\Omega; \nu(x) \cdot m(x) > 0\}.$$

Theorem 1

Let ω and $\mathcal{O}$ be neighborhoods of Γ_0. Assume that $a,\ c,\ d \in L^\infty(0, T; L^s(\Omega))$, with $s > 2$ for $N \in \{1,\ 2\}$ and $s \geq N$ for $N \geq 3$. Let $b \in L^\infty(Q)$, and suppose that there exists $b_0 > 0$ such that $b(x, t) \geq b_0$ for almost every (x, t) in $\mathcal{O} \times (0, T)$.

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For every $T > 2R_1$, there exists a positive constant C and a cut-off function $r \in \mathcal{D}^2((0, T))$ such that for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $(v^0, v^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, one has the observability estimate:

$$E(0) \leq C \int_0^T r \int_\omega (|u_t|^2 + |u|^2)\, dxdt$$

for the corresponding solution pair (u, v) of the adjoint system.

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It follows from this theorem and Lions' HUM that our initial system is exactly controllable with $h = ((ru_t)_t - ru)$ as a control.

Some literature

- Dáger (2006), $\Omega = (0, 1)$, $T \geq 4$, $b = -1_{\mathcal{O}}$, all other *l.o.t* vanish. Proved a weaker estimate; see Theorem 2 in the sequel.

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- Alabau-Leautaud (2012), $c = b$, $d = a$ are smooth enough, and $||b||_\infty$ is small, ω and $\mathcal{O}$ may have empty intersection, and both satisfy
  (GCC) [Bardos-Lebeau-Rauch, 1988, 1992]: every ray of geometric optics enters ω, (resp. $\mathcal{O}$) in a time less than T.
  But the controllability time blows up as the norm of the coupling function b goes to zero; this is not natural. One would expect the controllability cost to blow up as the coupling goes to zero, but not the controllability time.

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1. No smallness assumption is made on the potentials.
2. The controllability time is the same as for a single wave equation.
3. It seems that the coerciveness assumption on the coupling function b cannot be dropped; in particular, if b is zero, then one cannot estimate v in terms of u.
4. One may fairly wonder whether the observability estimate in Theorem 1 may be replaced with

$$E(0) \leq C \int_0^T r \int_\omega |u_t|^2 \, dxdt.$$

But as noted in the case of a single wave equation, that estimate is false in general, but holds under some constraints on the potential.

Proof of Theorem 1: key ideas

- Energy estimates show

$$E(0) \leq C \int_{Q_0} \{|u_t|^2 + |\nabla u|^2 + |v|^2\} \, dxdt,$$

where Q_0 is an appropriate subset of Q.

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where Q_0 is an appropriate subset of Q.

- Duyckaerts-Zhang-Zuazua + Fu-Yong-Zhang Carleman estimates show

$$\int_{Q_0} (|u_t|^2 + |\nabla u|^2 + |v|^2)\, dxdt \leq e^{-\mu\lambda} E(0) + C \int_0^T r^2 \int_{\omega_0} |v|^2\, dxdt \\ + C \int_0^T r \int_{\omega} (|u_t|^2 + |u|^2)\, dxdt$$

for every large enough $\lambda > 0$, and some fixed $\mu > 0$.

- Using a localizing arguments, one derives

$$\int_0^T r^2 \int_{\omega_0} |v|^2 \, dxdt \leq C \int_0^T r \int_\omega (|u_t|^2 + |u|^2) \, dxdt + e^{-\mu\lambda} E(0).$$

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- Choosing λ large enough, one gets:

$$E(0) \leq C \int_0^T r \int_\omega (|u_t|^2 + |u|^2) \, dxdt.$$

Theorem 2.

Let ω, $\mathcal{O}$, a, d and s be as in Theorem 1, and suppose that $b \in L^\infty(0, T; L^s(\Omega))$, $c \in L^\infty(Q)$, and there exists $b_0 > 0$ such that $b(x, t) \geq b_0$ for almost every (x, t) in $\mathcal{O} \times (0, T)$.

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For every $T > 2R_1$, there exists a positive constant C_0 such that for all $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, and $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$, one has the observability estimate:

$$\widehat{E}(0)^2 \leq C_0 \left(\int_0^T \int_\omega |u|^2 \, dxdt \right) (\widehat{E}_u(0) + E_v(0))$$

for all solution pair (u, v) of the adjoint system.

Proof of Theorem 2: Main ideas

Step 1. Prove the energy estimates

- $$\widehat{E}(t) \leq C_0 \widehat{E}(\tau), \qquad \forall \tau, t \in [0, T],$$

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- $$\widehat{E}(t) \leq C_0 \widehat{E}(\tau), \qquad \forall \tau, t \in [0, T],$$

- $$\int_{T_0}^{T_0'} h\widehat{E}(t)\, dt \leq C_0 \int_{Q_0} \{|u|^2 + |v|^2\}\, dxdt,$$

where h is an appropriate cut-off function.

Step 2. Derive from Step 1

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Step 3. Duyckaerts-Zhang-Zuazua Carleman estimate yields

$$\int_{Q_0} (|u|^2 + |v|^2)\, dxdt \leq e^{-\mu\lambda}\widehat{E}(0) + C_0 \int_0^T \int_\omega |u|^2\, dxdt + C_0 \int_0^T r^2 \int_{\omega_0} |v|^2\, dxdt,$$

for some constant $\mu > 0$, and every large enough λ.

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for some constant $\mu > 0$, and every large enough λ.

Step 4. Use a localizing argument to obtain:

$$\int_0^T r^2 \int_{\omega_0} |v|^2\, dxdt \leq C_0 \widetilde{E}(0)^{\frac{1}{2}} \left(\int_0^T \int_\omega |u|^2\, dxdt \right)^{\frac{1}{2}},$$

with $\widetilde{E}(0) = \widehat{E}_u(0) + E_v(0)$.

Let $a,\ b,\ c,\ d \in L^s(\Omega)$, with s as in Theorem 1. Let ω, $\mathcal{O}$, be as in Theorem 1, and suppose that there exists $b_0 > 0$ such that $b(x) \geq b_0$ for almost every x in $\mathcal{O}$.

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Further assume that either:

$$a \geq 0, \quad d \geq 0, \quad 2a - |b + c| \geq 0, \text{ and } 2d - |b + c| \geq 0, \text{ a.e. } x \in \Omega$$

or else

$$a \geq 0, \quad d \geq 0, \text{ a.e. } x \in \Omega, \quad 1 - C_s^2 |b + c|_s > 0, \text{ and } \lambda_0^2 - |b + c|_s > 0,$$

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where λ_0^2 is the first eigenvalue of the operator $-\Delta$ under Dirichlet boundary conditions, and C_s denotes the best constant in the Sobolev inequality:

$$||w||^2_{\frac{2s}{s-2}} \leq C_s^2 \int_\Omega |\nabla w(x)|^2\, dx, \quad \forall w \in H_0^1(\Omega).$$

Theorem 3

Assume the conditions just stated. For every $T > 2R_1$, there exists a positive constant C_0 such that for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $(v^0, v^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, one has the observability estimate:

$$(E_u(0) + E_v(0))^2 \leq C_0 \left(\int_0^T \int_\omega |u_t|^2 \, dxdt \right) (E_u(0) + \check{E}_v(0))$$

for all solution pair (u, v) of the adjoint system, and where $2\check{E}_v(0) = ||v^0||^2_{H^2(\Omega)} + ||v^1||^2_{H_0^1(\Omega)}$.

Sketch of the proof of Theorem 3

For this proof, we shall use Theorem 2, and the following result

Lemma

Let a, b, c, and d be given as in Theorem 3. Then there exists a positive constant $C_0 = C_0(\Omega, b+c)$ such that

$$\begin{aligned} &|| - \partial_i(b_{ij}(x)\partial_j u) + au + bv||^2_{H^{-1}(\Omega)} + || - \partial_i(b_{ij}(x)\partial_j v) + cu + dv||^2_{H^{-1}(\Omega)} \\ &\geq C_0 \int_\Omega \{b_{ij}(x)\partial_j u \partial_i u + b_{ij}(x)\partial_j v \partial_i v\}\, dx, \quad \forall u, v \in H^1_0(\Omega). \end{aligned}$$

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$$\|-\partial_i(b_{ij}(x)\partial_j u) + au + bv\|^2_{H^{-1}(\Omega)} + \|-\partial_i(b_{ij}(x)\partial_j v) + cu + dv\|^2_{H^{-1}(\Omega)}$$
$$\geq C_0 \int_\Omega \{b_{ij}(x)\partial_j u \partial_i u + b_{ij}(x)\partial_j v \partial_i v\}\, dx, \quad \forall u, v \in H^1_0(\Omega).$$

Set $\hat{w} = u_t$ and $\hat{z} = v_t$. Then these functions solve the system

$$\begin{cases} \hat{w}_{tt} - \partial_i(b_{ij}(x)\partial_j \hat{w}) + a\hat{w} + b\hat{z} = 0 \text{ in } Q \\ \hat{z}_{tt} - \partial_i(b_{ij}(x)\partial_j \hat{z}) + c\hat{w} + d\hat{z} = 0 \text{ in } Q \\ \hat{w} = 0, \quad \hat{z} = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ \hat{w}(0) = u^1 \in L^2(\Omega); \quad \hat{w}_t(0) = \partial_i(b_{ij}(x)\partial_j u^0) - au^0 - bv^0 \in H^{-1}(\Omega) \\ \hat{z}(0) = v^1 \in H^1_0(\Omega); \quad \hat{z}_t(0) = \partial_i(b_{ij}(x)\partial_j v^0) - cu^0 - dv^0 \in L^2(\Omega). \end{cases}$$

Introduce the following energy associated with that system

$$\widehat{E}_{\hat{w},\hat{z}}(t) = \widehat{E}_{\hat{w}}(t) + \widehat{E}_{\hat{z}}(t) \quad \forall t \in [0, T].$$

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Thanks to Theorem 2, one has:

$$\widehat{E}_{\hat{w},\hat{z}}(0)^2 \leq C_0 \left(\int_0^T \int_\omega |\hat{w}|^2 \, dxdt \right) (\widehat{E}_{\hat{w}}(0) + E_{\hat{z}}(0)).$$

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Hence

$$(E_u(0) + E_v(0))^2 \leq C_0 \left(\int_0^T \int_\omega |u_t|^2 \, dxdt \right) (E_u(0) + \check{E}_v(0)).$$

Theorem 4

Suppose that the hypotheses of Theorem 3 hold. For every $T > 2R_1$, there exists a positive constant $C = C(\Omega, \omega, \mathcal{O}, T, N, s, a, b, c, d)$ such that for all $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, and $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$, one has the observability estimate:

$$\check{E}_u(0) + E_v(0) \leq C \int_0^T r \int_\omega \{|u_t|^2 + |u_{tt}|^2\}\, dxdt.$$

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2. Are the analogues of Theorems 1 to 4 valid for $\omega \cap \mathcal{O} = \emptyset$, assuming ω and $\mathcal{O}$ both satisfy the Bardos-Lebeau-Rauch geometric control condition (GCC), and no smallness assumptions are made on the couplings?
3. What about different principal operators? A similar result holds for two wave equations coupled in cascade internally when the two operators are proportional & Ω is a compact C^∞ manifold with no boundary; in particular it is shown by Dehman-Leautaud-Lerousseau that if $\omega \cap \mathcal{O}$ satisfies GCC, then:

$$\widehat{E}(u;0) + E_{-2}(v;0) \leq C \int_0^T \int_\omega |u(x,t)|^2 \, dxdt,$$

where $2E_{-2}(v;0) = ||v^0||^2_{H^{-2}(\Omega)} + ||v^1||^2_{H^{-3}(\Omega)}$.

1. Do we have $E(u;0) + \widehat{E}(v;0) \leq C \int_0^T r \int_\omega |u_t(x,t)|^2 \, dxdt$, with no smallness assumption on the couplings?
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4. What about boundary controllability?

Controllability

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuously differentiable function with

$$\limsup_{|s|\to\infty} \frac{|f(s)|}{|s|(\log|s|)^{\alpha}} = \beta_0,$$

for some $\beta_0 > 0$, and some $0 \leq \alpha < 3/2$. Consider now the controllability problem: Given $y^0,\ \tilde{y}^0 \in H_0^1(\Omega)$, and $y^1,\ \tilde{y}^1 \in L^2(\Omega)$; $q^0,\ \tilde{q}^0 \in L^2(\Omega)$, and $q^1,\ \tilde{q}^1 \in H^{-1}(\Omega)$; and $\xi \in L^2(Q)$, can we find a control $v \in L^2(0, T; L^2(\omega))$ such that the corresponding solution pair (y_0, q) of the cascade system:

$$\begin{cases} y_{0tt} - \Delta y_0 + f(y_0) = \xi + v\chi_\omega \text{ in } Q \\ q_{tt} - \Delta q + f'(y_0)q = 0 \text{ in } Q \\ y_0 = 0, \quad q = \frac{\partial y_0}{\partial \nu_B}\chi_{\Gamma_0} \text{ on } \Sigma = \partial\Omega \times (0, T) \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1, \quad q(0) = q^0; \quad q_t(0) = q^1 \text{ in } \Omega, \end{cases}$$

satisfies:

$$y_0(.,T) = \tilde{y}^0, \quad y_{0t}(.,T) = \tilde{y}^1, \quad q(.,T) = \tilde{q}^0, \quad q_t(.,T) = \tilde{q}^1 \text{ in } \Omega?$$

For this system we have the controllability result:

Theorem 5

Assume that ω is a neighborhood of Γ_0. For every $T > 2R_1$, and for all $y^0 \in H_0^1(\Omega)$, $y^1 \in L^2(\Omega)$, $q^0 \in L^2(\Omega)$ and $q^1 \in H^{-1}(\Omega)$, there exists a control $v \in L^2(0, T; L^2(\omega))$ such that

$$y_0(., T) = \tilde{y}^0, \quad y_{0t}(., T) = \tilde{y}^1, \quad q(., T) = \tilde{q}^0, \quad q_t(., T) = \tilde{q}^1 \text{ in } \Omega.$$

To prove Theorem 5, we're going to follow the classical algorithm for solving control problems for nonlinear distributed systems:

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To prove Theorem 5, we're going to follow the classical algorithm for solving control problems for nonlinear distributed systems:

1. linearize the control problem,
2. solve the linear control problem,
3. use a fixed-point theorem to derive the controllability of the nonlinear problem from that of the linearized system.

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- Fu-Yong-Zhang (2007), Carleman estimates, hyperbolic equations,
- Duyckaerts-Zhang-Zuazua (2008), improved Carleman estimates, allows $\limsup_{|s|\to\infty} \dfrac{f(s)}{s(\log|s|)^{\alpha}} = 0, \quad 0 \leq \alpha < 3/2.$

The linear controllability problem

Set

$$g(s) = \begin{cases} (f(s) - f(0))/s, \text{ if } s \neq 0 \\ f'(0), \text{ if } s = 0. \end{cases}$$

Let $w \in L^\infty(0, T; L^2(\Omega))$. Set
$a(x,t) = g(w(x,t)), \quad b(x,t) = f'(w(x,t))$. The nonlinear controlled cascade system may be linearized as:

$$\begin{cases} y_{0tt} - \Delta y_0 + a(x,t) y_0 = -f(0) + \xi + v\chi_\omega \text{ in } Q \\ q_{tt} - \Delta q + b(x,t) q = 0 \text{ in } Q \\ y_0 = 0, \quad q = \frac{\partial y_0}{\partial \nu_B} \chi_{\Gamma_0} \text{ on } \Sigma \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1; \quad q(0) = q^0; \quad q_t(0) = q^1 \text{ in } \Omega \end{cases}$$

We shall find a control v so that:

$$y(T) = \tilde{y}^0; \quad y_t(T) = \tilde{y}^1, \quad q(T) = \tilde{q}^0; \quad q_t(T) = \tilde{q}^1 \text{ in } \Omega.$$

To this end, introduce the adjoint system:

$$\begin{cases} p_{tt} - \Delta p + b(x,t)p = 0 \text{ in } Q \\ \\ z_{tt} - \Delta z + a(x,t)z = 0 \text{ in } Q \\ p = 0, \quad z = \frac{\partial p}{\partial \nu_B}\chi_{\Gamma_0} \text{ on } \Sigma \\ p(T) = p^0; \quad p_t(T) = p^1, \quad z(T) = z^0; \quad z_t(T) = z^1 \text{ in } \Omega \end{cases}$$

For $(p^0, p^1) \in H_0^1(\Omega) \times L^2(\Omega)$, we have $p \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, and $z \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$. For every $t \in [0,T]$, define the energy

$$E(p;t) = \frac{1}{2}\left(||p_t(.,t)||^2_{L^2(\Omega)} + \int_\Omega |\nabla p(x,t)|^2\, dx \right).$$

Thanks to Lions' H.U.M, the linear controllability problem will be solved once we prove the following observability estimate:

Proposition

Let ω be a neighborhood of Γ_0, and let $T > 2R_1$. Let $\varepsilon > 0$ with $(N-2)\varepsilon < 4$. There exists

$$K_\varepsilon = \exp\left[C_\varepsilon(1 + ||a||_{\infty,l_\varepsilon}^{\frac{2}{3-2\theta_\varepsilon}} + ||b||_{\infty,l_\varepsilon}^{\frac{2}{3-2\theta_\varepsilon}})\right]$$

such that for all $(p^0, p^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and all $(z^0, z^1) \in L^2(\Omega) \times H^{-1}(\Omega)$:

$$E(p; T) + \widehat{E}(z; T) \leq K_\varepsilon \int_0^T \int_\omega |z(x,t)|^2 \, dxdt,$$

where $C_\varepsilon = C_\varepsilon(\varepsilon, \Omega, \omega, T, b_{ij}) > 0$, $l_\varepsilon = 2 + 4\varepsilon^{-1}$, $\theta_\varepsilon = \varepsilon N/(4 + 2\varepsilon)$, and $||.||_{\infty,r} = ||.||_{L^\infty(0,T;L^r(\Omega))}$.

Proof of Proposition: key elements

Step 1. Establish the energy estimate

$$E(p;t) \leq E(p;s)\exp\left(C_\varepsilon\left(1+||b||_{\infty,I_\varepsilon}^{\frac{1+\theta_\varepsilon}{2}}\right)|t-s|\right), \quad \forall s,t \in [0,T].$$

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Step 2. Use the Duyckaerts-Zhang-Zuazua (boundary) Carleman estimate and Step 1 to derive the boundary observability estimate

$$E(p;T) \leq e^{C_\varepsilon(1+||b||_{\infty,l_\varepsilon}^{\frac{2}{3-2\theta_\varepsilon}})} \int_0^T r^2 \int_{\Gamma_0} \left|\frac{\partial p(\gamma,t)}{\partial \nu_B}\right|^2 d\gamma dt.$$

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Step 3. Use a localizing argument to derive the partial estimate

$$E(p;T) \leq K_\varepsilon \int_0^T \int_\omega |z(x,t)|^2\, dxdt.$$

Step 4. Use the Duyckaerts-Zhang-Zuazua internal observability estimate to get

$$\widehat{E}(z;T) \leq K_\varepsilon \int_0^T \int_\omega |z(x,t)|^2\, dxdt + C(\Omega,T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial p(\gamma,t)}{\partial \nu_B} \right|^2 d\gamma dt.$$

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Step 5. Use Lions'inequality

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial p(\gamma,t)}{\partial \nu_B} \right|^2 d\gamma dt \leq K_\varepsilon E(p;T),$$

in Step 4, and combine the result with Step 3 to get the claimed estimate. □

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!