

# Temperature Gradient

## Derivation of $dT/dr$

In Modern Physics it is shown that the pressure of a photon gas in thermodynamic equilibrium (the radiation pressure  $P_R$ ) is given by the equation

$$P_R = \frac{1}{3}aT^4, \quad (1)$$

with  $a$  the radiation constant, which is related to the Stefan-Boltzman constant,  $\sigma$ , and the speed of light in vacuum,  $c$ , via the equation

$$a = \frac{4\sigma}{c} = 7.565731 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}. \quad (2)$$

Hence, the radiation pressure gradient is

$$\frac{dP_R}{dr} = \frac{4}{3}aT^3 \frac{dT}{dr}. \quad (3)$$

Referring to our our discussion of opacity, we can write

$$\frac{dF}{dr} = -\langle\kappa\rangle\rho F, \quad (4)$$

where  $F = F(r)$  is the net outward flux (integrated over all wavelengths) of photons at a distance  $r$  from the center of the star, and  $\langle\kappa\rangle$  some properly taken average value of the opacity. The flux  $F$  is related to the luminosity through the equation

$$F(r) = \frac{L(r)}{4\pi r^2}. \quad (5)$$

Considering that pressure is force per unit area, and force is a rate of change of linear momentum, and photons carry linear momentum equal to their energy divided by  $c$ ,  $P_R$  and  $F$  obey the equation

$$P_R = \frac{1}{c}F(r). \quad (6)$$

Differentiation with respect to  $r$  gives

$$\frac{dP_R}{dr} = \frac{1}{c} \frac{dF}{dr}. \quad (7)$$

Combining equations (3), (4), (5) and (7) then gives

$$-\frac{1}{c}\langle\kappa\rangle\rho \frac{L(r)}{4\pi r^2} = \frac{4}{3}aT^3 \frac{dT}{dr}, \quad (8)$$

which finally leads to

$$\frac{dT}{dr} = -\frac{3\langle\kappa\rangle\rho}{16\pi ac} \frac{1}{T^3} \frac{1}{r^2} L(r). \quad (9)$$

Equation (9) gives the local (i.e. at a distance  $r$  from the center of the star) temperature gradient that is required to maintain the local energy flow  $L(r)$ . The latter will be determined by how much energy is generated inside the sphere of radius  $r$  (through the quantity  $\epsilon(r)$  discussed in class). Inside a star, the ratio  $T^3/\rho$  does not vary enormously, and the required  $dT/dr$  will be determined primarily by the opacity  $\langle\kappa\rangle$  and the luminosity  $L(r)$ . Large values of either of these two may require large  $dT/dr$ 's, and the question then arises whether the temperature gradient can increase without bound. The answer is it cannot. Once the temperature gradient that is needed to carry energy outward by radiative diffusion were to exceed (in absolute value) the *adiabatic* temperature gradient, large macroscopic cells of gas will develop that rise a certain distance, cool, and sink again, thereby heating higher layers in the star. We then have transport of energy through moving cells of gas, a process called convection. The criterion (developed by Karl Schwarzschild in 1906) to decide whether energy will be carried by convection is, if the absolute value of the local temperature gradient exceeds the absolute value of the local adiabatic temperature gradient, the layer becomes convective and energy will be carried outward through convective motions of blobs of gas. In a convective layer, the actual temperature gradient will no longer be given by equation (9), but by the adiabatic temperature gradient itself, given by the equation

$$\left(\frac{dT}{dr}\right)_{\text{adiabatic}} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr}, \quad (10)$$

with  $\gamma$  the ratio of specific heats at constant pressure and at constant volume,

$$\gamma = \frac{c_P}{c_V}. \quad (11)$$

For a monatomic, ideal gas,  $\gamma$  is equal to 5/3.

Note that calculating the mean opacity  $\langle\kappa\rangle$  is complicated. To a good first approximation, this quantity is proportional to the density  $\rho$  and inversely proportional to the temperature raised to a power 3.5. As a result, you will sometimes see the following approximation for the mean opacity:

$$\langle\kappa\rangle = \kappa_0 Z(1 + X)\rho T^{-3.5}, \quad (12)$$

with  $\kappa_0$  a constant,  $Z$  the fraction by mass of heavy elements,  $X$  the fraction by mass of hydrogen,  $\rho$  the density and  $T$  the temperature. Equation (12) is called a *Kramers opacity law*, named after H. A. Kramers who developed this expression in 1923 using classical physics.

### Adiabatic Temperature Gradient

Consider a gas at pressure  $P$  and temperature  $T$ , occupying a volume  $V$ . From thermodynamics, for an adiabatic change of state, it follows that

$$PV^\gamma = \text{constant}, \quad (13)$$

with  $\gamma$  given by (11). Logarithmic differentiation gives

$$\frac{dP}{P} + \gamma \frac{dV}{V} = 0, \quad (14)$$

or,

$$\frac{dP}{P} - \gamma \frac{d\rho}{\rho} = 0. \quad (15)$$

The equation of state of an ideal gas,

$$P = \frac{\rho}{\mu m_p} kT, \quad (16)$$

gives, after logarithmic differentiation,

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}. \quad (17)$$

Equations (15) and (17) give equation (10), after some algebra.