The Golden Radius in Balanced Atmospheric Flows

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ABSTRACT

In gradient-balanced, cyclonic flow around low pressure systems, a golden radius exists where R_G, the gradient-wind Rossby number, is $\phi^{-1} = 0.618\,034$, the inverse golden ratio. There, the geostrophic, cyclostrophic, and inertia-circle approximations to the wind all produce equal magnitudes. The ratio of the gradient wind to any of these approximations is ϕ^{-1}. In anomalous (anticyclonic) flow around a low, the golden radius falls where $R_G = -\phi = -1.618\,034$, and the magnitude of the ratio of the anomalous wind to any of the two-term approximations is ϕ. In normal flow, the golden radius marks the transition between more-nearly cyclostrophic and more-nearly geostrophic regimes. In anomalous flow, it marks the transition between more-nearly cyclostrophic (anticyclonic) and inertia-circle regimes. Over a large neighborhood surrounding the golden radius, averages of the geostrophic and cyclostrophic winds weighted as ϕ^{-2} and ϕ^{-3} are good approximations to the gradient wind. In high pressure systems R_g, the geostrophic Rossby number, must be in the range $0 > R_g \geq -1/4$, and the pressure gradient cannot produce inward centripetal accelerations. An analogous radius where $R_g = -\phi^{-3}$ plays a role somewhat like that of the golden radius, but it is much less interesting.

1. Introduction

The relationship between winds and pressure in circular atmospheric pressure systems is well represented using the gradient-wind approximation (e.g., Willoughby 1990). The Rossby number (V/fr, where V is a characteristic wind speed, f is the Coriolis parameter, and r is a characteristic horizontal length) characterizes the balance of forces. In high-Rossby-number lows (strong winds, small spatial scales, or weak background rotation), gradient balance reduces to cyclostrophic balance. When the Rossby number is small (weaker winds, larger spatial scales, or stronger background rotation), the wind is approximately geostrophic (Holton 2004, 62–68; Dutton 1976, 311–314). The transition between these two regimes occurs where the magnitudes of the Rossby numbers based upon cyclostrophic and geostrophic winds, R_C and R_g, are one. As shown below, when this criterion is met, the gradient-wind Rossby number is ϕ^{-1}, the inverse golden ratio, in normal flow and $-\phi$ in anomalous flow. At the golden radius, errors between the gradient and geostrophic or cyclostrophic wind and local approximations to the gradient wind are readily formulated in terms of powers of ϕ. In high pressure systems, a somewhat analogous radius where $R_g = -\phi^{-3}$ has a similar, but not so enchanting, role.

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2. What is the golden ratio?

The golden ratio (e.g., Livio 2002) is the ratio between a side and a chord connecting two adjacent sides of a regular pentagon (Fig. 1a). In Fig. 1a, all of the shaded isosceles triangles are similar. If the sides of the pentagon are 1, equating the ratio of one of the equal sides of the top-left triangle (1) to the base ($\phi - 1$) to the ratio of one of the equal sides (ϕ) of the large central triangle to its base (1) yields

$$\frac{1}{\phi - 1} = \frac{\phi}{1}. \tag{1}$$

Clearing the fraction and rearranging produces a quadratic equation for ϕ:

$$\phi^2 - \phi - 1 = 0, \tag{2}$$

which has solution

$$\phi = \frac{1}{2}(1 \pm \sqrt{5}) = 1.618\,034, \ -0.618\,034. \tag{3}$$

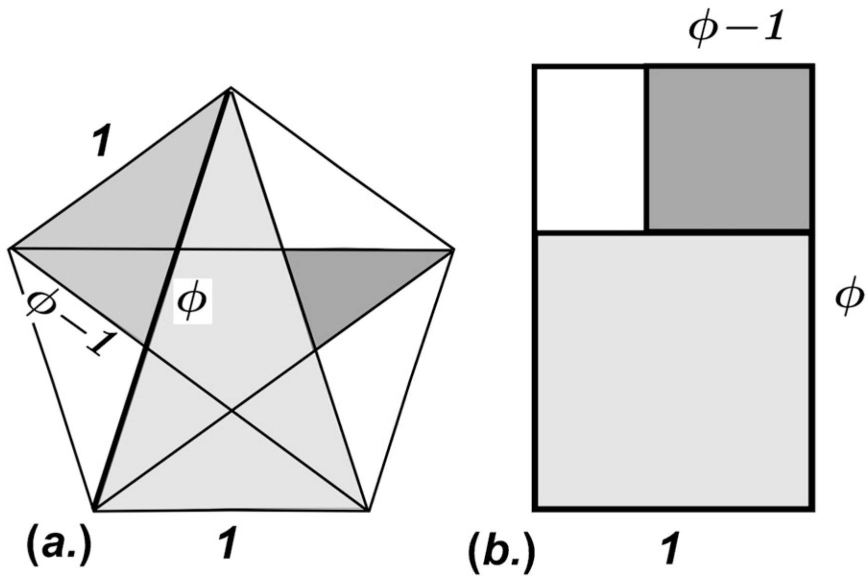

FIG. 1. (a) The geometry of a regular pentagon, illustrating the derivation of the golden ratio by similar triangles. (b) A golden rectangle subdivided into a square and a smaller golden rectangle, which is subdivided again.

From (2), powers of ϕ may be written $\phi^2 = \phi + 1$, $\phi^3 = 2\phi + 1$, $\phi^4 = 3\phi + 2, \ldots$. Similarly, negative powers are $\phi^{-1} = \phi - 1$, $\phi^{-2} = 2 - \phi$, $\phi^{-3} = 2\phi - 3$, $\phi^{-4} = 5 - 3\phi, \ldots$.

The golden ratio is ubiquitous in the visual arts (Livio 2002), turns up occasionally in numerical analysis (Press et al. 1986), is used to set architectural proportions (Le Corbusier 1954), and even occupies entire chapters of adventure novels (Brown 2003, 117–128). The golden rectangle, with proportions 1: ϕ, is thought to be most aesthetically pleasing (Fig. 1b). It may be subdivided into a square with unit sides and a smaller golden rectangle with sides 1 and $\phi^{-1} = \phi - 1$. This rectangle may be subdivided into successively smaller squares and golden rectangles, each a factor of ϕ^{-1} smaller than its predecessor. "Sightings" of ϕ are a cottage industry among enthusiasts. Many are genuine; some are fortuitous, or simply spurious. For example, the conversion of statute miles to kilometers is 1.609 344 mi km^{-1}, a value within 0.44% of ϕ.

3. The golden radius

Gradient balance provides a good approximation to the wind in atmospheric flows where the friction is small and the advective and local rates of change are so slow that the pressure gradient balances the Coriolis and centripetal accelerations required for the air to follow a circular path in a rotating reference frame:

$$\frac{V_G^2}{r} + fV_G = \frac{1}{\rho}\frac{\partial p}{\partial r}. \tag{4}$$

Here, V_G is the gradient wind, r is the radius from the system center, f is the Coriolis parameter, ρ is the air density, and p is pressure. The present analysis focuses on the tangential component of the wind in circular pressure distributions. The sign convention used here is that the radius is always positive and the sign of the wind determines the sense of rotation. If one takes the terms in (4) two at a time, the first term on the left and the term on the right yield cyclostrophic balance, the second term on the left and the term on the right yield geostrophic balance, and the terms on the left yield an inertia-circle balance:

$$V_C = \pm\sqrt{\frac{r}{\rho}\frac{\partial p}{\partial r}}, \quad V_g = \frac{1}{f\rho}\frac{\partial p}{\partial r}, \quad V_I = -fr. \tag{5}$$

Cyclostrophic flow is possible only in low pressure systems, where it may be cyclonic or anticyclonic, geostrophic flow (V_g) is cyclonic ($V_g > 0$) around lows and anticyclonic ($V_g < 0$) around highs, and inertia circles are always anticyclonic.

The geostrophic and cyclostrophic Rossby numbers are $R_g = V_g/fr$ and $R_C = V_C/fr$. Here, R_g and R_C take on the sign of V_g or V_C since $r > 0$. The relationship between them is

$$R_g = \frac{1}{fr}\left(\frac{1}{f\rho}\frac{\partial p}{\partial r}\right) = \frac{V_C^2}{f^2 r^2} = R_C^2. \tag{6}$$

Similarly, the relationship between the geostrophic and cyclostrophic winds is

$$V_g = \frac{1}{f\rho}\frac{\partial p}{\partial r} = \frac{1}{f}\frac{V_C^2}{r} = R_C V_C. \tag{7}$$

Rewriting (4) with the pressure gradient expressed in terms of the geostrophic or cyclostrophic winds yields a dimensional relation similar to Fultz's (1991) nondimensional one:

$$\frac{V_G^2}{r} + fV_G = fV_g = \frac{V_C^2}{r}. \tag{8}$$

Solving the quadratic for V_G, and multiplying both the numerator and denominator by $\frac{1}{2}(-1 \pm \sqrt{1+4R_g})$ yields

$$V_G = \frac{V_g}{\frac{1}{2}\left(1 \pm \sqrt{1+4R_g}\right)} = \frac{V_C R_C}{\frac{1}{2}\left(1 \mp \sqrt{1+4R_C^2}\right)} = \frac{V_C}{\frac{1}{2}\left(R_C^{-1} \pm 2\sqrt{1+\frac{1}{4}R_C^{-2}}\right)}. \tag{9}$$

See Petterssen (1956, 60–65) for an alternate derivation. Either expression in (9) is accurate at any r. In normal flow around low pressure systems (positive root, cyclonic), V_G approaches V_C as R_C or R_g becomes large and approaches V_g as R_C or R_g becomes small. In anomalous flow (negative root, anticyclonic), V_G approaches $-V_C$ for large R_C or R_g, and V_G approaches $V_I = -fr$ for small R_C or R_g. In high pressure systems, we have $0 > R_g \geq -1/4$. The latter inequality is essential for vortex stability. Both normal and anomalous flows are anticyclonic. When $|R_g|$ is small in non-anomalous flow around a high, V_G approaches V_g. In anomalous flow, V_G approaches V_I.

In low pressure systems at the golden radius r_{Au} where $R_g = R_C = 1$, $V_g = V_C = fr_{Au}$, normal gradient balance in (9) yields $V_G = \phi^{-1} V_g = \phi^{-1} V_C$. Anomalous gradient balance with $R_g = 1$ yields $V_G = -\phi V_g = -\phi|V_C|$. These results are consistent with Fultz (1991), who recognized that $R_G = 0.618$ and -1.618 when $R_g = \pm 1$, but did not explicitly identify the golden ratio, and with the apparently well-known existence of a radius where $V_g = V_C = fr$ (information online at http://en.wikipedia.org/wiki/Balanced_flow). The salient result here is that r_{Au}, given by the implicit equations $V_G(r_{Au})/fr_{Au} = \phi^{-1}$ in normal flow and $V_G(r_{Au})/fr_{Au} = -\phi$ in anomalous flow, marks the outward transition from approximately cyclostrophic to approximately geostrophic or inertia-circle flow. Since the actual wind in the free atmosphere is often nearly in gradient balance, these relations can be used to find r_{Au}, for example in winds observed as functions of radius by aircraft flying in hurricanes.

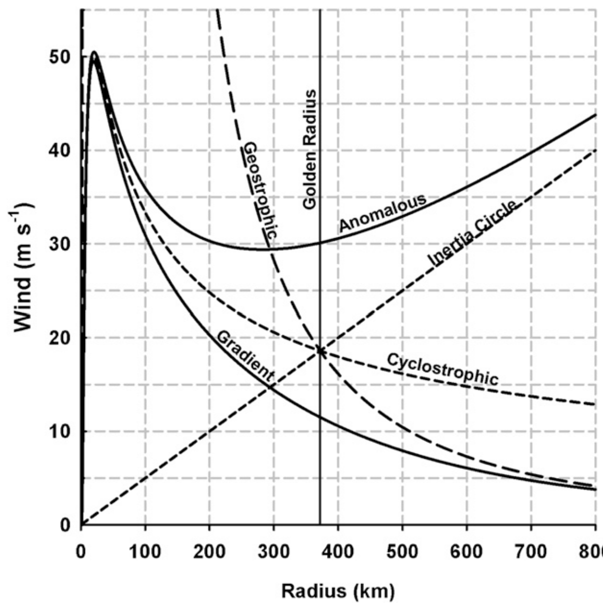

FIG. 2. Geostrophic, cyclostrophic, inertia-circle, gradient, and anomalous gradient winds as functions of radius in a hurricane-like vortex, illustrating the relative magnitudes of the different approximations at the golden radius. Note that the inertia-circle and anomalous winds are anticyclonic, although they are plotted as positive values for comparison.

Figure 2 illustrates the radial variation of the wind for a hurricane-like vortex (Holland 1980) at 20°N with 50 m s^{-1} maximum wind 20 km from the center. The golden radius lies at r_{Au} = 372 km. It was located by calculating R_g as a function of radius and scanning the values for 0.618. At r_{Au}, the cyclostrophic, geostrophic, and inertial winds are all 18.6 m s^{-1}. The normal gradient wind is 11.5 m s^{-1}, and the anomalous wind is 30.1 m s^{-1}. Inward from r_{Au}, R_C increases, and the cyclostrophic approximation improves; outward from r_{Au}, R_g decreases, and the geostrophic approximation improves. The differences between the gradient and geostrophic (or cyclostrophic) winds at r_{Au} are $V_G - V_g = (\phi^{-1} - 1)V_g = [(\phi - 1) - 1]V_g = (\phi - 2)V_g = -\phi^{-2}V_g$ for normal flow, and $V_G + V_g = (-\phi + 1)V_g = -\phi^{-1}V_g$ for anomalous flow. Since $V_g = V_C$ at r_{Au}, the error made by these approximations is larger at r_{Au} than at any other radius.

Two weighted averages of the geostrophic and cyclostrophic winds, $V_{23} = \phi^{-2}V_C + \phi^{-3}V_g = (2 - \phi)V_C + (2\phi - 3)V_g$ and $V_{32} = \phi^{-3}V_C + \phi^{-2}V_g = (2\phi - 3)V_C + (2 - \phi)V_g$, are exactly equal to non-anomalous V_G at the golden radius, where $V_C = V_g$. They also approximate V_G well in a surrounding neighborhood (Fig. 3). The former relation, which weights the cyclostrophic wind more heavily, is more accurate when $r < r_{Au}$; the latter, which weights the geostrophic wind more heavily, is more accurate when $r > r_{Au}$. For the case shown in Fig. 3, the

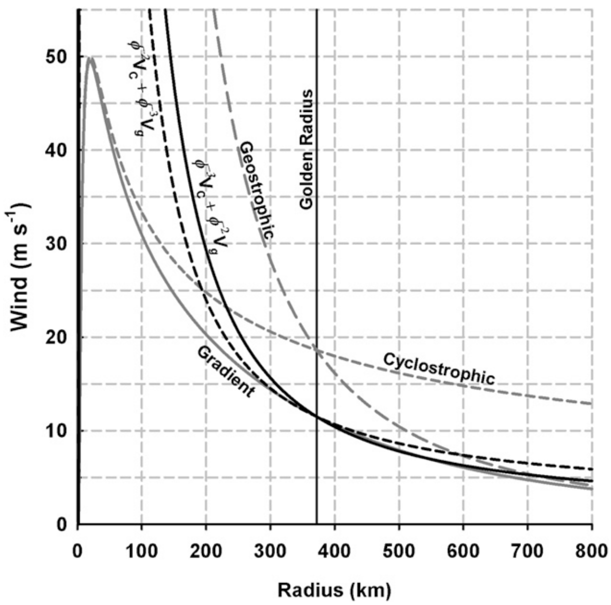

FIG. 3. Approximation of the gradient wind for the vortex shown in Fig. 2, using weighted averages of the geostrophic and cyclostrophic winds. The weighting factors are ϕ^{-2} and ϕ^{-3}.

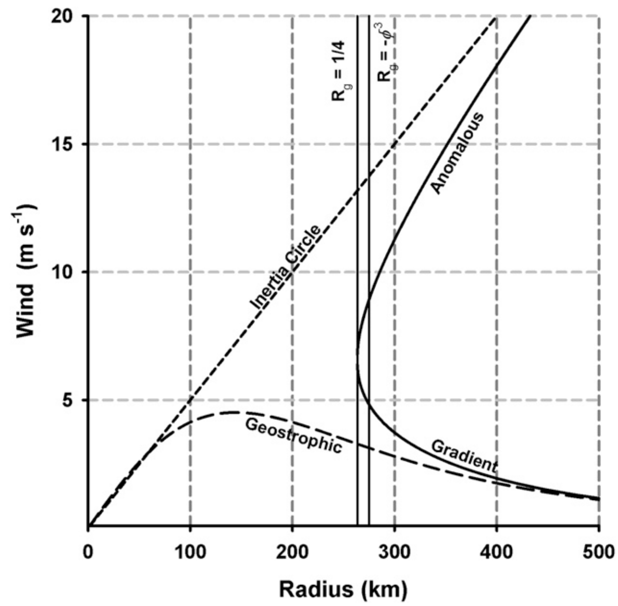

FIG. 4. Geostrophic, inertia-circle, gradient, and anomalous gradient winds as functions of radius in an idealized high pressure system, illustrating the relative magnitudes of the different approximations at the "copper radius," where $R_g = \phi^{-3}$. Note that all wind components illustrated are anticyclonic.

better of these approximations is closer to the gradient wind than either the geostrophic or cyclostrophic wind for 200 km $\leq r \leq$ 800 km. The corresponding relation for anomalous flow would use $-V_C$ and $-fr$ weighed by ϕ^{-1} and 1.

The dynamics are much different in high pressure systems (Fig. 4). The pressure profile used is $p(r) - p_\infty = (p_c - p_\infty)a^2/(a^2 + r^2)$, where p_∞ is the pressure outside the high, p_c is the pressure at the center, $p_\infty - p_c = 1$ hPa, $a =$ 250 km is the profile size parameter, and the latitude is 45°, where $f = 1 \times 10^{-4}$. Highs cannot support cyclostrophic winds, and their geostrophic Rossby numbers must lie in the range $0 > R_g \geq -1/4$. The situation shown would be unstable near the center, where $R_g < -1/4$. Where $R_g = -1/4$, both normal and anomalous winds are anticyclonic, and $V_G = 2V_g$. The largest inverse integer power of the golden ratio $\leq 1/4$ is $\phi^{-3} = 0.2279$. Since it is so close to the critical value, geostrophic Rossby numbers $= -\phi^{-3}$ should occur near the centers of high pressure systems, if at all. Unlike r_{Au} in lows, this "copper" radius in highs, where $V_g(r_{Cu}) = -\phi^{-3}fr_{Cu} = -0.2279fr_{Cu}$, does not mark a change of flow regimes, although $R_g = -1/4$ certainly does. The normal anticyclonic winds at r_{Cu} are ϕ times the geostrophic winds and the anomalous anticyclonic winds are ϕ^2 times the geostrophic wind.

4. Conclusions

The golden radius, where the Rossby number based upon the non-anomalous gradient wind in low pressure systems is equal to the inverse golden ratio ($\phi^{-1} =$ 0.618 034), marks the transition from the inner vortex, where the cyclostrophic approximation is more accurate, to the outer vortex, where the geostrophic approximation is more accurate. For anomalous flow, the gradient-wind Rossby number at the golden radius is equal to $-\phi =$ $-1.618\,034$, and the golden radius marks the transition from approximately cyclostrophic, anticyclonic flow to approximately inertia-circle flow. At the golden radius, the winds computed from the cyclostrophic, geostrophic, and inertia-circle approximations are all equal. It is possible to approximate the gradient winds in a neighborhood of the golden radius as sums of the cyclostrophic and geostrophic winds weighted with ϕ^{-2} and ϕ^{-3} for normal flow and the anticyclonic cyclostrophic and inertia-circle winds weighted with ϕ^{-1} and 1 for anomalous flow. In high pressure systems, cyclostrophic winds are impossible. At a radius where the geostrophic Rossby number $= -\phi^{-3}$ in highs, the anticyclonic non anomalous, and anomalous gradient winds are ϕ and ϕ^2 times the geostrophic wind.

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