Objective: To introduce the equations of motion and describe the forces that act upon the Atmosphere

Reading: Read pp 18-26 in Chapter 1 of *Houghton & Hakim*

Problems: Work 1.1, 1.8, and 1.9 on pp. 26 & 27 at the end of Chapter 1 of *Holton & Hakim*. They are due in class Wednesday 08SEP17

Motions in the Atmosphere are governed by:

- Newton's second law, $F = ma$, force is equal to mass times acceleration.
- First law of thermodynamics, conservation of energy
- Mass conservation for various substances
- Ideal gas law relating air density, temperature, and pressure

The first two relations are first-order partial differential equations in ***Initial-Value Form (IVF)*** with the first-order, time derivative of a single dependant variable on the left and values or spatial partial derivatives on the right In IVF, the equations are straightforward to integrate numerically in time.

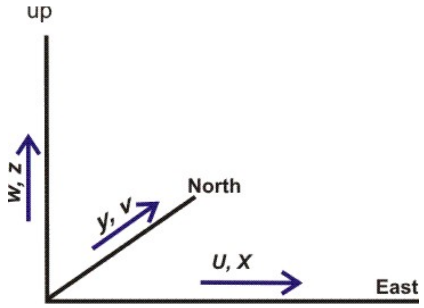
The mass conservation laws and gas law are ***diagnostic*** relations inasmuch as they do not contain time derivatives.

Newton's Second Law in meteorological coordinates:

$$\rho\frac{Du}{Dt} = \text{Forces in the } x \text{ direction}$$

$$\rho\frac{Dv}{Dt} = \text{Forces in the } y \text{ direction}$$

$$\rho\frac{Dv}{Dt} = \text{Forces in the } z \text{ direction}$$



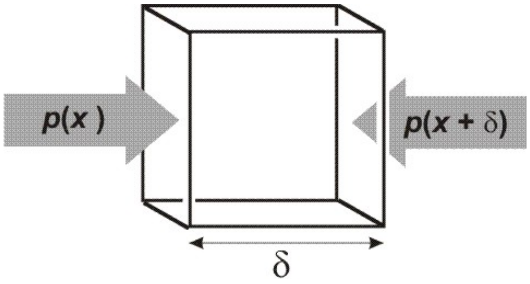
Here $D(\,)/Dt$ are Lagrangian derivatives that follow moving "parcels" of air; *u*, *v*, and *w* are velocity components in the *x*, *y*, and *z* directions; *t* is time and ρ is the density of air. For now, interpret D( )/Dt as being the same as d( )/dt, but it will take on another meaning.

We represent distances and velocities as vectors, for example $\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$ or $\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$ the carets ("hats") over *i*, *j* and *k* indicate that they are unit vectors that point in the *x*, *y*, and *z* directions.

What about the forces on the right side of Newton's second law?

1. **Gravity** $= -g\rho\hat{k}$, where g = 9.8 m s^{-1}. Gravity always acts in the –*z* direction.

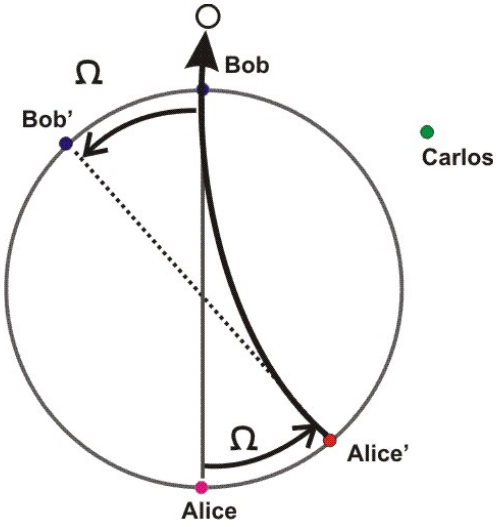
2. **Pressure Gradient Force:** Consider a cubic parcel of air with each side = δ. The sketch at the right shows how the difference in pressure between the right and left sides of the parcel result is a net force. The force on the left side is $p(x)\delta^2$ and that on the right is is $p(x+\delta)\delta^2$. The mass of the parcel is $\rho\delta^3$, so that the net force in the x direction divided by the

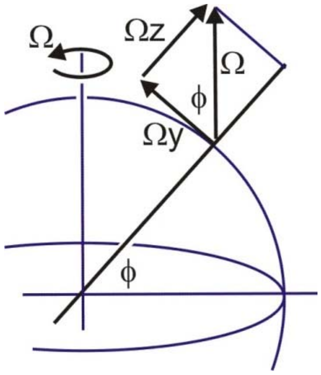


mass is $\delta^2[p(x+\delta)-p(x)]/\rho\delta^3$, or $[p(x+\delta)-p(x)]/\rho\delta = \rho^{-1}\partial p/\partial x$. Applying the same argument in the y and z directions yields the pressure gradient acceleration $= \rho^{-1}\left(\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right) = \rho^{-1}\nabla p$

where $p = p(x, y, z, t)$, $\partial/\partial x$ is a partial derivative that implies differentiation with respect to x only, and similarly for $\partial/\partial y$ and $\partial/\partial z$.

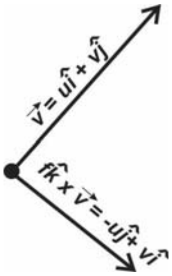
3. Coriolis Force: $= \rho\left[\hat{i}(2\Omega\sin\varphi)v - \hat{j}(2\Omega\sin\varphi)u\right]$ + small (with respect to gravity) vertical terms. A game of catch between Alice and Bob, both of whom in a rotating reference frame illustrates how this apparent force arises. After Alice throws the ball toward Bob, both players rotate to positions Alice' and Bob'. From the perspective of Carlos, who is in a non-rotating reference frame the ball flies straight from Alice's original position to Bob's original position. From the perspectives of the players the ball misses Bob' because of an apparent force that deflects it to the right of its intended position.





Except at the poles of the spherical planet the Earth's rotation will have a horizontal component, which gives rise to vertical apparent forces that are << g. Similarly, since $w \ll (u, v)$, we neglect the Coriolis terms induced by vertical motions. Thus only the component of the Earth's rotation around the local vertical is significant. As the figure at the left shows, it is $f = 2\Omega\sin\varphi$, where φ is the latitude. We call f the Coriolis parameter.

At the pole, the Coriolis parameter is $2\Omega = (2\pi/86400\text{ s}) = 1.4533 \times 10^{-4}\text{ s}^{-1}$.



At 20° latitude the Coriolis parameter is $2\Omega\sin 20^\circ = 0.497 \times 10^{-4}\text{ s}^{-1}$
$\approx 5 \times 10^{-5}\text{ s}^{-1}$
At 45° latitude the Coriolis parameter is $2\Omega\sin 45^\circ = 1.028 \times 10^{-4}\text{ s}^{-1}$
$\approx 1 \times 10^{-4}\text{s}^{-1}$
The Coriolis force acts perpendicular to the horizontal velocity $= -fu\hat{j} + fv\hat{i}$
We will analyze the Coriolis force in more detail later on.

4. Friction: Written in various ways—
   a. $-\rho\mu(u\hat{i} + v\hat{j})$, where μ is the coefficient of friction

b. $-\rho C_D \sqrt{u^2+v^2}\,(u\hat{i}+v\hat{j})$, where C$_D$ is the drag coefficient

c. $\rho K\left[\hat{i}\left(\frac{\partial u^2}{\partial x^2}+\frac{\partial u^2}{\partial y^2}+\frac{\partial u^2}{\partial z^2}\right)+\hat{j}\left(\ldots+\frac{\partial v^2}{\partial z^2}\right)+\hat{k}\left(\ldots+\frac{\partial w^2}{\partial z^2}\right)\right]$, where *K* is the kinematic viscosity.

Friction is important only near the surface, where the z derivatives of the horizontal wind components predominate.

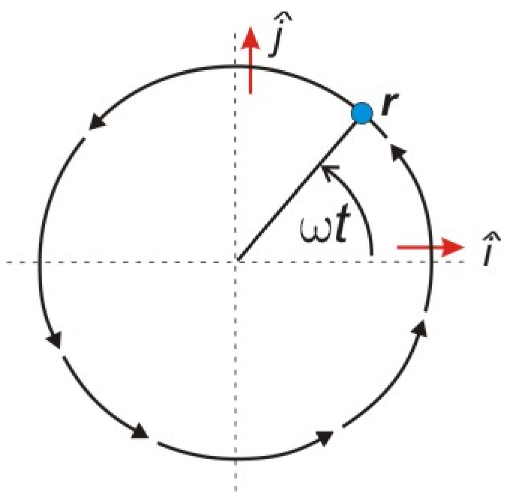
So the components of Newton's second law for the atmosphere, called the ***Momentum Equations***, can be written:

$$\frac{Du}{Dt}=fv-\frac{1}{\rho}\frac{\partial p}{\partial x}+K\left(\ldots+\frac{\partial u^2}{\partial z^2}\right)$$

$$\frac{Dv}{Dt}=-fu-\frac{1}{\rho}\frac{\partial p}{\partial y}+K\left(\ldots+\frac{\partial v^2}{\partial z^2}\right)$$

$$\frac{Dw}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g$$

Let's look at one more kind of motion: Circular motion. Imagine an object rotating in a circular path such that.



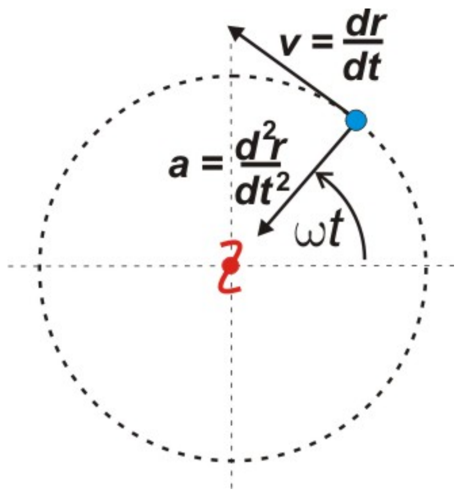
$\mathbf{x}=x\hat{i}=\hat{i}r\cos\omega t$

$\mathbf{y}=y\hat{j}=\hat{j}r\sin\omega t$

$\mathbf{r}=r\hat{r}=x\hat{i}+y\hat{j}$, and

$r=\sqrt{x^2+y^2}$

Here bold face ***x*** and ***y*** are the vector components of the object's position. Each is the product of scalar magnitudes, *x* and *y* with unit vectors $\hat{i}$ and $\hat{j}$ pointing the positive directions along the abscissa and ordinate, respectively. The position vector **r**, with magnitude *r* points along the time varying $\hat{r}$ direction. It is the vector sum of ***x*** and ***y***.



The time derivative of r is:

$$\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{x}}{dt}+\frac{d\mathbf{y}}{dt}=\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}\equiv u\hat{i}+v\hat{j}\equiv\mathbf{v}$$
$$=(\omega r\sin\omega t)\hat{i}+(-\omega r\cos\omega t)\hat{j}$$
$$=\omega r(\hat{i}\sin\omega t-\hat{j}\cos\omega t)$$

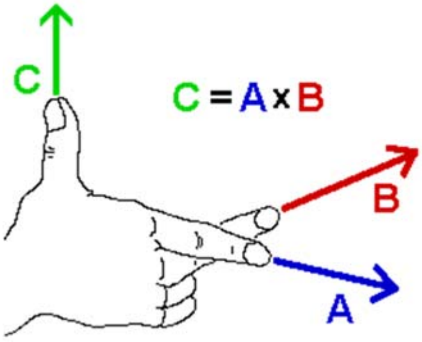
Note that the motion is perpendicular to the position vector and its magnitude is proportional to the product of the position vector's magnitude with the rotation frequency.

$$\begin{aligned}\frac{d\mathbf{v}}{dt}=\frac{d\mathbf{u}}{dt}+\frac{d\mathbf{v}}{dt}&=\frac{du}{dt}\hat{i}+\frac{dv}{dt}\hat{j}\equiv\mathbf{a}\\&=(-\omega^2 r\cos\omega t)\hat{i}+(-\omega^2 r\sin\omega t)\hat{j}\\&=-\omega^2 r(\hat{i}\cos\omega t+\hat{j}\sin\omega t)=-\omega^2\mathbf{r}=-(v^2/r)\hat{r}\end{aligned}$$

Here, again the acceleration's magnitude is proportional to the product of the rotation frequency with the magnitude of the velocity vector. Since the velocity vector is perpendicular to the position vector, the acceleration vector points toward the center of rotation, in the opposite direction from the position vector. This is a reasonably rigorous derivation of the ***Centripetal Acceleration*** required for an object to move around a circular path.

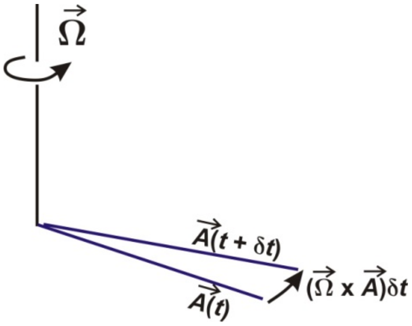
It is also an example of a vector product, sometimes called the cross product. For two vectors ***A*** and ***B*** that make an angle θ with each other, their vector product ***C*** is formally defined as.

$$\mathbf{C}=\mathbf{A}\times\mathbf{B}=AB\sin\theta$$



C is oriented perpendicular to the plane defied by ***A*** and ***B***. Its direction is determined by using the right-hand rule. If you place your right hand on the plane defiened by the vectors such that your index finger points in the A direction and your middle finger points in the B direction, your thumb will point in the ***C*** direction.

If we define $\boldsymbol{\omega}=\omega\hat{k}$ where k is the unit vector perpendicular to the plane defined by $\hat{i}$ and $\hat{j}$ the ***v*** = ***ω*** x ***r*** and ***a*** = ***ω*** x ***ω*** x ***r*** = −ω²***r.***



Let's extend this argument to a vector that may be changing with time in a reference frame that rotates with the Earth at a constant angular velocity Ω = 2π/24h. In such a reference frame, any vector obeys:

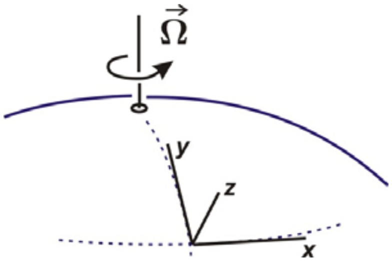
$$\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{FIX}}=\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{ROT}}+\mathbf{\Omega}\times\mathbf{A}$$

And the second derivative is:

$$\begin{aligned}\left(\frac{d^2\mathbf{A}}{dt^2}\right)_{\mathrm{FIX}}&=\left(\left(\frac{d}{dt}\right)_{\mathrm{ROT}}+\mathbf{\Omega}\times\right)\left[\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{ROT}}+\mathbf{\Omega}\times\mathbf{A}\right]\\&=\left(\frac{d^2\mathbf{A}}{dt^2}\right)_{\mathrm{ROT}}+2\mathbf{\Omega}\times\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{ROT}}+\mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{A}\end{aligned}$$

But $\mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{A}=$ the ***Centripetal Force*** due to rotation of the reference frame. As we showed above, it is always perpendicular to the axis of rotation and nearly constant because ***A*** is primarily made up of $R_E$cos ϕ. The centripetal force is << g and the vector sum of the centripetal force and gravity is everywhere perpendicular to the surface defined by sea level. Thus, we can write $g' = g + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{A}$ but we will drop the primes henceforth. With these changes Newton's second law emerges as:

$$\frac{D\mathbf{v}}{Dt} + 2\mathbf{\Omega}\times\mathbf{v} = -\frac{1}{\rho}\nabla p - g'\mathbf{k}$$



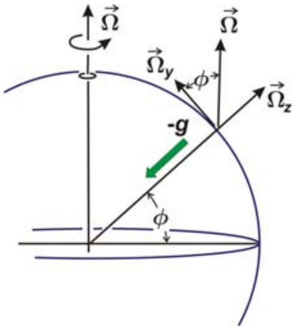
As before, we place a Cartesian coordinate system tangent to the Earth's surface.

Thus *u*, *v*, and *w* become spherical components of the velocity in a shallow layer above sea level.

The non-Cartesian, geometrical terms in the equations are more-or-less negligible for large-scale meteorological motions.

Expanding the Coriolis acceleration term,

$$2\mathbf{\Omega}\times\mathbf{v} = 2\Omega[(w\cos\varphi - v\sin\varphi)\hat{i} + (u\sin\varphi)\hat{j} - (u\cos\varphi)\hat{k}]$$



The vertical component is << that g, and (as shown subsequently) w << (u, v) so that ,

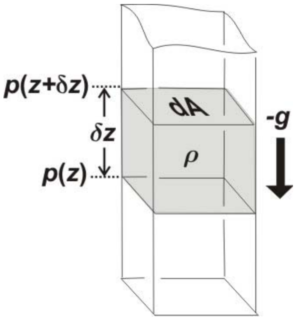
$$2\mathbf{\Omega}\times\mathbf{v} \cong 2\Omega\sin\varphi(-v\hat{i} + u\hat{j}) = f\,\hat{k}\times\mathbf{v}$$

Where, as before, $f = 2\Omega\sin\phi = 2*(2\pi/86400\text{ s})*\sin\phi$ is the **Coriolis Parameter**. Its units are s^{-1}.

Hydrostatic Law

Let's look at the vertical momentum equation again.

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - g$$



For synoptic-scale, middle latitude motions the vertical velocity, *w*, is 1-10 cm s^{-1}. Synoptic scale motions require about a day (86400 s) to change appreciably. Thus, the vertical acceleration is about 0.1 m s^{-1}/(86400 s) $\simeq$ 10-6 m s^{-2} << 9.8 m s^{-2}, and the two terms on the right must be equal.

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g, \text{ or } \frac{\partial p}{\partial z} = -g\rho \quad \text{(Hydrostatic law)}$$

The gas law for dry air, $p/\rho = p\alpha = R_d T$ relates pressure, *p*, temperature, *T*, and density, ρ,or specific volume, α, such tha $\alpha = \rho^{-1}$. R_d is the gas constant for dry air = 287 J kg^{-1} K^{-1}. More about the gas law later.

$$\frac{\partial p}{\partial z} = -g\rho = -\frac{gp}{R_d T}, \text{or } \frac{1}{p}\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}\ln p = -\frac{g}{R_d T}$$

For the special case where *T* is constant with height, we can integrate the equation easily to get *p*(*z*),

$$\ln p = -\frac{gz}{R_d T} + \text{const.}$$

At the surface, where $z = 0$, $p = p_0$

$$\ln p_0 = -0 + \text{const.}$$

Subtracting the second equation from the first:

$$\ln \frac{p}{p_0} = -\frac{gz}{R_d T}$$

Taking the exponential of both sides,

$$p(z) = p_0 \exp\left\{-\frac{gz}{R_d T}\right\}. = p_0 e^{-z/H}$$

We call the quantity *H* the scale height. It is the height difference over which the pressure (or density) in an isothermal atmosphere decreases by a factor of $e^{-1} = 0.3679$. A typical vertically averaged temperature in the troposphere is 255K, so

$$H = \frac{R_d T}{g} = \frac{(287\text{m}^2\text{s}^{-1}\text{K}^{-1})(255\text{K})}{9.8\text{m s}^{-1}} = 7467.8\text{m} = 7.5\text{km}$$

This is the same vertical scale that we talked about in Lecture 0