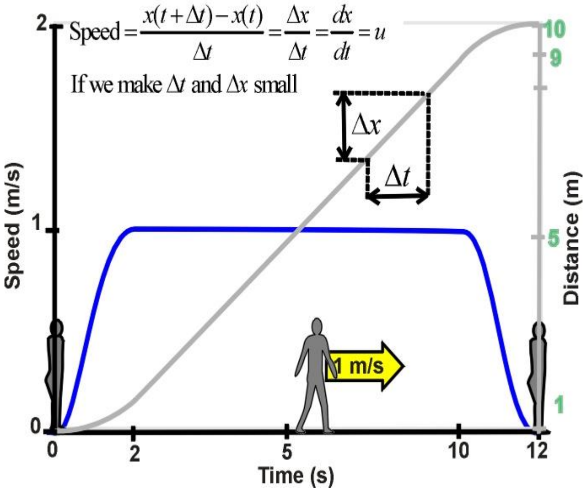

Objectives: To review essential mathematics for Meteorological Dynamics

Reading: Notes

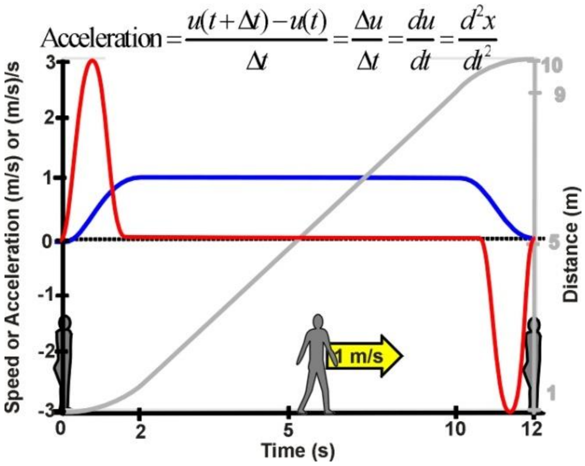
Problems: Handout sheet

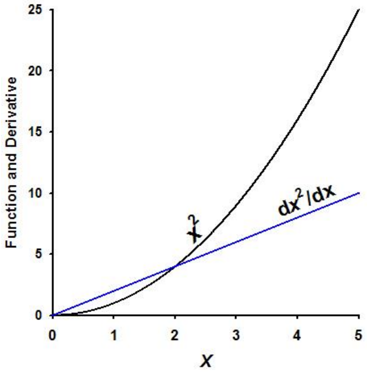
About Derivatives

Derivatives represent the rate of change of a known algebraic function. Most powerful physical laws—including those that describe the atmosphere---are formulated in terms of derivatives. I call this the ***Newtonian Paradigm***, because Isaac Newton (and Gottfried Leibnitz) first used derivatives to define the laws of gravitation and mechanics. They also invented The ***Calculus*** to compute derivatives and antiderivatives (integrals).



Consider a professor strolling across the campus. The derivative of his position is the difference between his position now and his position a short time ago divided by the time interval that elapsed between his occupying the two positions. The derivative is defined to be that ratio in the limit that the time interval goes to zero. The Prof's speed is the derivative of position with respect to time. If the change of position has a direction associated with it---that is, if it is a vector---it is called velocity. His acceleration is the derivative of speed with respect to time, or the second derivative of his position with respect to time. There are well-defined ways of going from known functions to their derivatives or from the derivative back to the function. For now, we'll focus on derivatives.





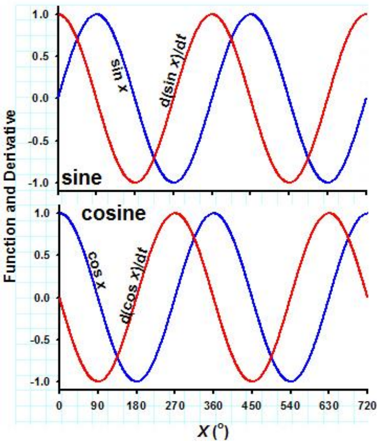
For example, the derivative of x^2:

$$\begin{aligned}\frac{d(x^2)}{dx} &= \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= 2x + \Delta x \simeq 2x\end{aligned}$$

Derivatives of x^n all work the same way. For any n, $d(x^n)/dx = nx^{n-1}$. This recipe works even if n is negative or not a whole number. For example $dx^{½}/dx = ½(1/x^{½})$. And of course dx^0/dt = d (constant) /dt = 0

Derivatives of sines and cosines

They are derived in the same way as for powers, but using trigonometric sum-of-angle formulas:



d (sin x)/dx = cos x

d (cos x)/dx = −sin x

and for second derivatives:

d^2 (sin x)/dx^2 = d (cos x)/dx= −sin x

d^2 (cos x)/dx^2 = d (sin x)/dx = −cos x

Note that differentiating either the sine or cosine twice returns the negative of the original function.

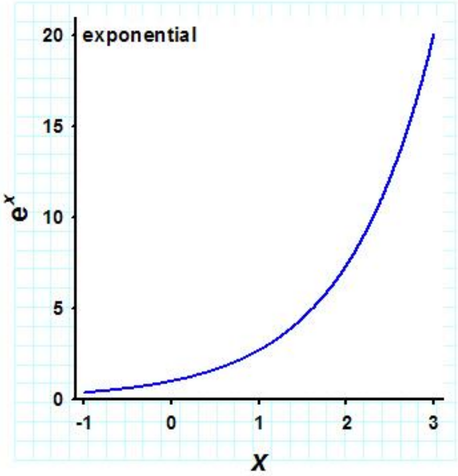
Exponential Function

$e^x = \exp\{x\} = (2.71828\ldots)^x$

e is Euler's number, or the base of the natural logarithms

$e^x * e^y = e^{x+y}$

$(e^x)^n = e^{nx}$



$de^x/dx = e^x$

Natural logarithms are the inverse functions for exponentials

$\ln(e^x) = x$

$\ln x + \ln y = \ln xy$

$n\ln x = \ln x^n$

$d(\ln x)/dx = 1/x$

Some Other Calculus Rules

Derivative of a sum

$d(x + y)/dt = dx/dt + dy/dt$

Derivative of a product

$dxy/dt = x\, dy/dt + (dx/dt)\, y$

Derivative of a quotient

$D(x/y)/dt =(-x/y^2)dy/dt + (1/y)(dx/dt)$

Chain rule for a function of a function:

$d\, f(g(t))/dt = (df/dg)*(dg/dt)$

Example: $d(\exp\{-kx^2\})/dx =\exp\{-kx^2\}\, (-2kx)$, or $d(\sin kx)/dx = k \cos kx$

Partial derivatives

In partial derivatives one differentiates only by the specified variable where it appears explicitly in a function of several variables, keeping all others constant. For example:

Let $g(x,y,z) = xy^2z^3$, then

$$\frac{\partial g}{\partial x} = y^2z^3, \quad \frac{\partial g}{\partial y} = 2xyz^3, \text{ and}$$

$$\frac{\partial g}{\partial z} = 3xy^2z^2.$$

Summary 1

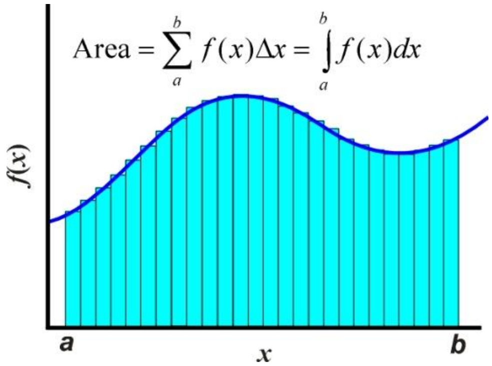
- Derivatives represent rates of change of functions.
- Easily remembered recipes for calculating them from functions:
- $d\, x^n/dx = nx^{n-1}$
- $d(\cos x)/dx = -\sin x$; $d(\sin x)/dx = \cos x$
- $d\, e^x/dx = e^x$; $d(\ln x)/dx = 1/x$
- Chain rule
- Rules for sums, quotient, and products

Integration:

Integration is the opposite of differentiation—the anitderivative. Often you can recognize a function as the derivative of another function.

$$F(x) = \int \frac{dF(x)}{dx} dx + \text{constant, or}$$
$$f(x) = \frac{d}{dx}\left(\int f(x) dx + \text{constant} \right)$$

Where *f*(*x*) is any function of *x*



Integrals are also a way to calculate the area under a curve.

We can turn the derivatives that we already know into integrals easily

$$\int x^n dx = \frac{1}{n+1}\int (n+1)x^n dx = \frac{x^{n+1}}{n+1} + \text{constant}$$
$$\int \sin x \; dx = -\cos x + \text{constant}$$
$$\int \cos x \, dx = \sin x + \text{constant}$$
$$\int e^x dx = e^x + \text{constant}$$
$$\int \frac{dx}{x} = \ln x + \text{constant}$$

More about exponentials:

We define the square root of negative one as:

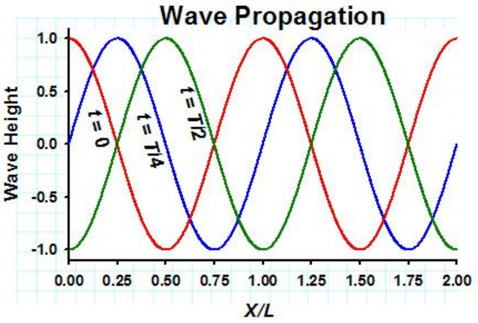
$$i \equiv \sqrt{-1}, \text{or } i^2 = -1$$

Numbers of the form *Yi* are called ***Imaginary***; Numbers of the form *X* + *Yi* (that is, with both real and imaginary parts) are called ***Complex***.

These definitions lead us to Euler's formula for imaginary exponentials:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler's formula is useful because it simplifies the algebra required to describe oscillations and wave propagation.



$$\begin{aligned} W &= \cos 2\pi(x/L - t/T) \\ &= \mathrm{Re}\left[\exp\{2\pi i(x/L - t/T)\}\right] \end{aligned}$$

Calculating derivatives with Euler's equation:

$$\frac{\partial W}{\partial t} = \mathrm{Re}\left[\frac{2\pi i}{T}W\right] = -\frac{2\pi}{T}\mathrm{Im}\left[W\right],$$

$$\frac{\partial W}{\partial x} = \mathrm{Re}\left[\frac{2\pi i}{L}W\right] = -\frac{2\pi}{L}\mathrm{Im}\left[W\right]$$

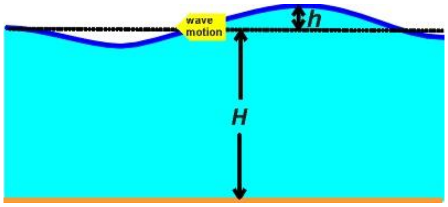
Define:

$\omega = \frac{2\pi}{T} =$ frequency, and $k = \frac{2\pi}{L} =$ wavenumber, then

$$\frac{\partial W}{\partial t} = \mathrm{Re}\left[i\omega W\right], \quad \frac{\partial W}{\partial x} = \mathrm{Re}\left[ikW\right]$$

Wave speed is $L/T = \omega/k$

We can use Euler's formula to solving (partial) differential equations , by writing $h(x,t) = H_o\exp\{i(kx - \omega t)\}$ and substituting into the partial differential equation (which we will derive later) that describes shallow-water waves:



$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2}$$

$$-\omega^2 h = gH(-k^2 h), \text{or}$$

$$\frac{\omega^2}{k^2} = \frac{(2\pi/T)^2}{(2\pi/L)^2} = \frac{L^2}{T^2} = (\text{wave speed})^2 = gH$$

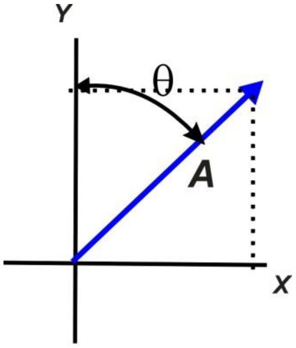
$$\text{wave speed} = \pm\sqrt{gH}$$

Here ω = 2π/T is the wave's ***Frequency*****,** and k = 2π/L is its ***Wavenumber*** in the x direction**.**

Summary 2

- Integration is the reverse of differentiation
- Areas under curves
- Constants of integration determined by limits
- Standard formulas
- Euler's formula $e^{ix} = \cos x + i \sin x$
- Used to solve partial differential equations for wave motion

Vectors



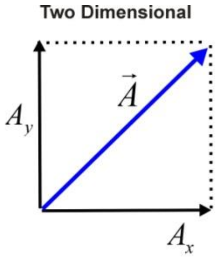
A VECTOR has both magnitude and direction. It is composed of components that lie along particular directions. In this illustration of a two dimensional vector, $A_y = A \cos \theta$ and $A_x = A \sin \theta$.

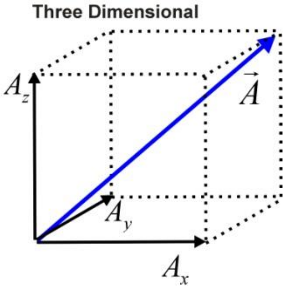
Vectors are different from SCALARS, which have only magnitude.

Meteorological and Vector Winds

Wind it the prototypical meteorological vector. Meteorologists reckon winds are as the direction *from* which the wind is blowing. In the illustration below, the wind blowing from the NNW. Vector winds are reckoned as the direction *toward* which the wind is blowing, SSE here. We also use degrees: 0 = N, 045 = NE, 090 = E, 180 = S, 270 = W, 315 = NW, etc.. The values in degrees that correspond to the Cardinal (N, S, E, W) and Ordinal (NW, SW, SE, NE) are the same for vector or meteorological directions, but vector direction = met direction + 180, subtract 360 if the answer is > 360.

Vector Components:





Two-dimensional vector components (*X* and *Y*): *X* component points EAST (in meteorology) and *Y* points NORTH. The vector is written $\mathbf{V}_2 = X\hat{i} + Y\hat{j} = X\mathbf{i} + Y\mathbf{j}$. Here **Bold Face** indicates a vecor quantity and **i**and **j**, and **k** are unit vectors pointing east and north, respectively.

Three dimensional vector components: *X* and *Y* point East and North, as before, and Z points up. Three dimensional vectors are written $\mathbf{V}_3 = X\hat{i} + Y\hat{j} + Z\hat{k} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, where **k** is the vertical unit vector. I use caret and bold-face notation interchangeably, and sometimes I write vectors as $\vec{V}$, for example in the figure on the left. For the most part I use caret/arrow notation on the whiteboard in class

Wind vectors are a special case:

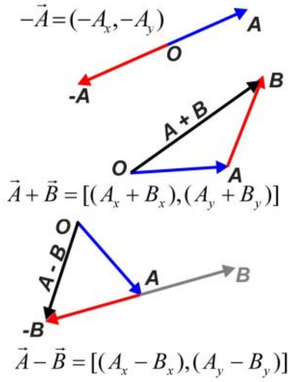
Two Dimensional:
$\mathbf{V} = u\mathbf{i} + v\mathbf{j}$
Three Dimensional:
$V = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$
Generally $w \ll u, v$

Vector magnitude and direction (reckoned anticyclonically from north) in terms of components:

$$\tan\theta = \frac{A_x}{A_y}$$

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2}$$

Adding and Subtracting Vectors:



The negative of a vector is the negative of its components. It has the same magnitude but opposite direction.

The sum of two vectors is the sum of their components. It can be calculated graphically by placing them end-to-end and connecting the start of the first vector with the end of the second.

Difference of two vectors is the sum the first with the negative of the second.

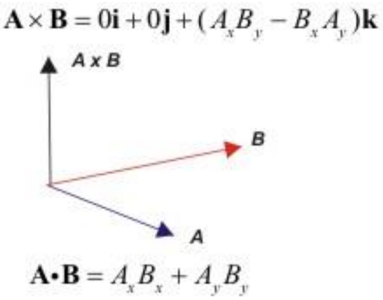
Vector Products:

The ***Dot Product*** (inner product) is computed by multiplying the corresponding components. It produces a scalar. It is used, for example, to represent work and energy in the atmosphere.

$$\begin{aligned}\mathbf{A}\bullet\mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\bullet(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

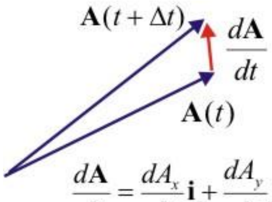
In the **Cross Product** (outer product) each component is the difference of the products of the normal components with sign switches. The resulting vector is oriented perpendicular to the plane defined by the vectors being multiplied. It is used, for example, to represent the Coriolis force.

The illustration at the right shows the cross product of two dimentional vectors in a plane. The three dimensional cross product is the determinant composed of the unit vectors (top row), the components of the first vector (middle row), and the components of the second vector. It is readily evaluated through expansion by minors.



$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z\end{vmatrix} \\ &= \mathbf{i}(A_yB_z - A_zB_y) - \mathbf{j}(A_xB_z - A_zB_x) + \mathbf{k}(A_xB_y - A_yB_x)\end{aligned}$$

Derivatives of Vectors:



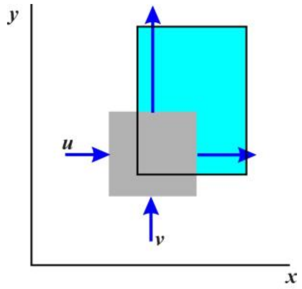
$$\frac{d\mathbf{A}}{dt} = \frac{dA_x}{dt}\mathbf{i} + \frac{dA_y}{dt}\mathbf{j}$$
$$= \boldsymbol{\omega} \times \mathbf{A} + \frac{d|\mathbf{A}|}{dt}$$

Derivative of a vector is a vector composed of the derivatives of its components

For any vector, its derivative can be broken down into:

- A rotation, represented as the cross product of a perpendicular angular velocity vector
- A stretching or shrinking of the magnitude of the original vector

Divergence:



$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

The divergence represents expansion (or contraction) of an element of air when the wind increases (or decreases) downstream. If the divergence is negative it is called ***Convergence***.

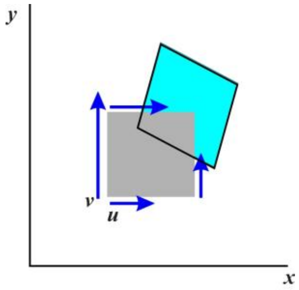
Divergence is a scalar

Horizontal divergence near the surface causes sinking motion in the Atmosphere above

Horizontal convergence near the surface causes rising motion in the Atmosphere above

A flow with zero divergence is said to be ***Nondivergent***

Curl



$$\nabla \times \vec{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The Curl represents rotation and distortion of an element of air when the wind increases cross stream

An example of the curl's use in meteorology is relative vorticity

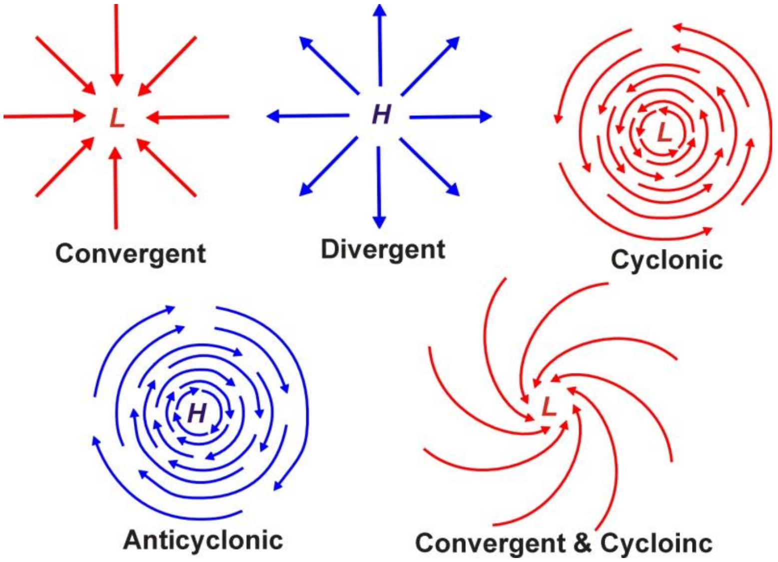
The sum of curl of the wind vector and the rotation of the Earth tends to stay constant following a parcel of air

But convergence tends to increase the curl and divergence tends to decrease it.

These ideas are central to understanding weather

A flow with zero curl is said to be ***Irrotatonal***

Examples of Flows:

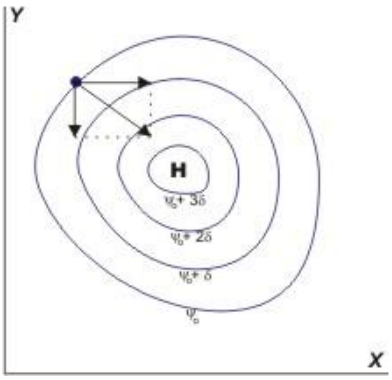


Gradient

The gradient represents the total spatial change of a scalar as a function of position.

It is the vector sum of the partial derivatives with respect to each of the coordinates times the unit vector pointing in that coordinate direction.

$$\nabla \psi = \frac{\partial \psi}{\partial x}\mathbf{i} + \frac{\partial \psi}{\partial y}\mathbf{j} + \frac{\partial \psi}{\partial z}\mathbf{k}$$



An example of the gradient is the pressure gradient force in the Momentum Equations.

Laplacian.

The Laplacian of a scalar is the scalar that results from taking divergence of its gradient

$$\nabla^2\psi = \nabla \bullet \nabla \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Functions that satisfy ***Laplaces' Equation***, $\nabla^2\psi = 0$, are said to be ***Harmonic***. A related equation, ***Poisson's Equation***, $\nabla^2\psi = f(x, y, z)$ plays a key role in several important meteorological problems As a preview of future analyses, consider what ψ would look like if f were products of sines and cosines of constants times *x*, *y* and *z*.

Summary 3

- Vector: Magnitude and direction
- Scalar: Magnitude only
- Project onto components along coordinate axes
- Vector wind = meteorological wind + 180^o
- Sums and differences: Add and subtract components
- Derivative; Derivative of components
- Inner (dot) product: Multiply parallel components (Scalar)
- Outer (cross) product: Differences of products of perpendicular components (Vector)
- Divergence and curl, like inner and outer products but using derivatives
- Divergence: expansion and contraction
- Curl: Rotation and some deformation
- Gradient point uphill for a function of spatial coordinates
- Laplacian is the divergence of the curl