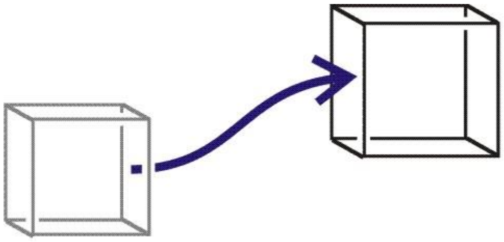
Objective: To introduce

- Eulerian and Lagrangian differentiation
- Equation of state
- Mass conservation
- Thermodynamic energy conservation

Reading: Read pp 28-30 and 42-53 in Chapter 2 of *Holton and Hakim*.

Problems: Work 2.1, 2.3, and 2.4 on p. 65, due Wednesday 05OCT15

Eulerian and Lagrangian Derivatives



What about those D( )/Dt time derivatives? They follow parcels moving with the air, but we can also calculate derivatives measured by an arbitrarily-moving probe, such as an aircraft. Special cases of the Lagrangian derivative are the time derivatives of the parcels' positions such that D*x*/D*t* = *u,* D*y*/D*t* = *v*, and D*z*/D*t* = *w*.

If we are measuring some quantity *q* with a moving probe, the values might change because *q* is varying with time locally or because the probe is moving to places where the values are different. Note that d*x*/d*t*, … denote arbitrary motion through the fluid, which may , or may not, be the same as the fluid motion.

$$\frac{dq}{dt}=\frac{\partial q}{\partial t}+\frac{\partial q}{\partial x}\frac{dx}{dt}+\frac{\partial q}{\partial y}\frac{dy}{dt}+\frac{\partial q}{\partial z}\frac{dz}{dt}.$$

A special case is the Largangian derivative, when the probe is attached to the moving air parcels and we include sources or sinks of *q* in addition to local time and spatial variations.

$$\frac{Dq}{Dt}=\frac{\partial q}{\partial t}+\frac{\partial q}{\partial x}\frac{Dx}{Dt}+\frac{\partial q}{\partial y}\frac{Dy}{Dt}+\frac{\partial q}{\partial z}\frac{Dz}{Dt}=Q$$

$$\frac{Dq}{Dt}=\frac{\partial q}{\partial t}+u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}+w\frac{\partial q}{\partial z}=Q$$

$$\frac{\partial q}{\partial t}+\mathbf{v}\bullet\nabla q=Q$$

The first term on the left is the ***Lagrangian*** derivative following the parcel.

In the middle expression the first term after the equal sign is the local, ***Eulerian,*** time derivative that represents the change with time at a fixed position.

The next three terms in the middle expression are ***Advection*** by the zonal, meridional, and vertical components of the wind.

The final term after the second equal sign represents ***Sources*** or ***Sinks*** of *q*.

As we have seen before, motions in the Atmosphere are governed by:

- Newton's second law, $F = ma$
- First law of thermodynamics, conservation of energy
- Mass conservation for various substances
- Ideal gas law relating air density, temperature, and pressure

The first three relations are prognostic, first-order partial differential equations in ***Initial-Value Form (IVF)***. The gas law is a ***diagnostic*** relation inasmuch as it does not contain time derivatives.

Newton's Second Law from last time with the derivatives expanded:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial y}$$

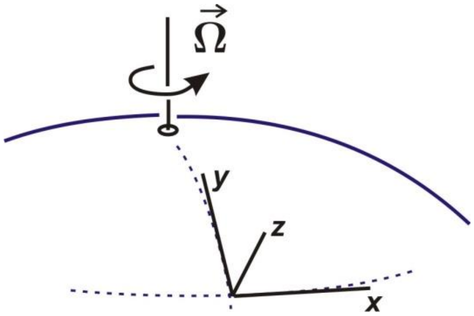
$$= \frac{\partial u}{\partial t} + \mathbf{v}\bullet\nabla u = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} + K\left(\ldots + \frac{\partial u^2}{\partial z^2}\right)$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial t} + \mathbf{v}\bullet\nabla v = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} + K\left(\ldots + \frac{\partial v^2}{\partial z^2}\right)$$

$$\frac{Dw}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial y}$$

$$= \frac{\partial w}{\partial t} + \mathbf{v}\bullet\nabla w = -\frac{1}{\rho}\frac{\partial p}{\partial x} - g$$

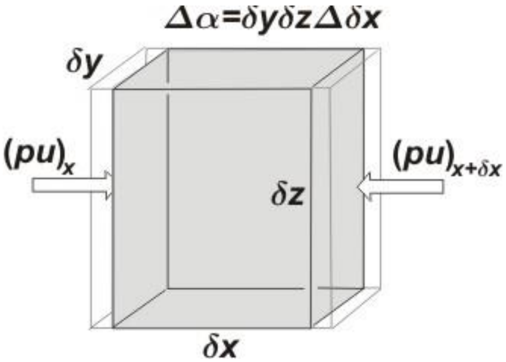


For example, a spatially uniform zonal flow may accelerate everywhere through the ∂u/∂t, (Eulerian term), or a steady zonal flow may accelerate downstream through the $u\partial u/\partial x$ (Advection) term. More generally all of the terms in the derivative are nonzero.

First Law of Thermodynamics:

$$c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt} = c_v\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right)$$

$$+p\left(\frac{\partial \alpha}{\partial t} + u\frac{\partial \alpha}{\partial x} + v\frac{\partial \alpha}{\partial y} + w\frac{\partial \alpha}{\partial z}\right) = J$$



T is absolute temperature; $c_v = 717$ J kg^{-1} K^{-1} is the specific heat of dry air at constant volume; p is pressure; J is the heating rate; and α is the specific volume = $1/\rho$ = 1/(density). This equation represents the balance between changes of internal energy c_vT, pressure work $p\alpha$ and heating, J. If $J = 0$, the thermodynamic process is said to be ***Adiabatic***, if J is nonzero it is ***Diabatic*** (heat flowing into or out of the parcel). In the pressure-work term, external forces push (or pull) against the force due to pressure times the area to change the volume of the parcel and do work.

Universal Gas Law

Relates the pressure, p_g, density, ρ_g, and temperature, T, of any gas with molecular weight m,

$$\frac{p_g}{\rho_g} = \frac{R^*}{m} T \,,$$

where R^* = 8314 J K^{-1} kmol^{-1} is the universal gas constant. When we apply this law to dry to dry air whose molecular mass is 28.97 kg kmol^{-1} (It's mostly N_2, molecular weight ≈28, ⅕ O_2, molecular weight ≈ 32, and < 1 % Ar, molecular weight ~ 40),

$$\frac{p_d}{\rho_d} = p_d \alpha_d = R_d T$$

Where R_d is the gas constant for dry air = 8314/28.97 = 287 J/(kg-K) and $\alpha_d = 1/\rho_d$ is the ***Specific Volume***. The Gas law in this form is sometimes called the ***Equation of State***.

The version of the gas law for water vapor alone (molecular mass 18.016 kg kmol^{-1}) is:

$$\frac{e}{\rho_v} = e\alpha_v = R_v T$$

Where R_v = 8314/18.016 = 461.9 J kg^{-1} K^{-1}, By Dalton's law from chemistry, the total pressure is made up of the partial pressure due to each gas acting independently, $p = p_d + e$. Since the gasses are mixed, they have the same temperature. Application of Dalton's law to the mixture of gasses making up dry air is how we got 28.97 for the effective molecular weight of dry air. It's not a simple average. Thus the density of the combination of dry air and vapor is:

$$\begin{aligned}
\rho = \rho_d + \rho_v &= \frac{p_d}{R_d T} + \frac{e}{R_v T} = \frac{p-e}{R_d T} + \frac{e}{R_v T} = \frac{p}{R_d T} - \frac{e}{R_d T} + \frac{e}{R_v T} \\
&= \frac{p}{R_d T}\left[1 - \frac{e}{p} + \frac{R_d e}{R_v p}\right] = \frac{p}{R_d T}\left[1 - \left(1 - \frac{R_d}{R_v}\right)\frac{e}{p}\right] \qquad , \\
&\cong \frac{p}{R_d T\left[1 + \left(1 - \frac{R_d}{R_v}\right)\frac{e}{p}\right]} = \frac{p}{R_d T_v}
\end{aligned}$$

Note that because the molecular mass of vapor is less than that for dry air $R_d / R_v = (R^*/m_d)/(R^* m_v) = m_v/m_d$ = 18.016/28.97 = 0.6219 and $1 - m_v/m_d$ = 0.378. Thus, the *Virtual Temperature*, T_v, is always warmer than the actual temperature. There is more to this relation…

Another form of the Thermodynamic Equation.

Differentiating the gas law for dry air:

$$\frac{D\alpha}{Dt} = \frac{D}{Dt}\left(\frac{R_d T}{p}\right) = \frac{R_d}{p}\frac{DT}{Dt} - \frac{R_d T}{p^2}\frac{Dp}{Dt}$$

Substituting into the First Law,

$$c_v \frac{DT}{Dt} + p\left(\frac{R_d}{p}\frac{DT}{Dt} - \frac{R_d T}{p^2}\frac{Dp}{Dt}\right) = J \, ;$$

$$(c_v + R_d)\frac{DT}{Dt} - \frac{R_d T}{p}\frac{Dp}{Dt} = c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt}$$

$$c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) - \alpha\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) = J \, ,$$

Where $c_p = c_v + R_d$ = 1005 J kg^{-1} K^{-1} is the specific heat at constant pressure, and $c_v = c_p - R_d$ = 718 J kg^{-1} K^{-1}.

Note that J kg^{-1} K^{-1} = kg m^2 s^{-2}/(kg K) = m^2 s^{-2} K^{-1}. Thus, specific heats have units of velocity squared per Kelvin.

Let's look at the special case of adiabatic (J = 0), steady vertical motion.

$$c_p\left(0 + w\frac{\partial T}{\partial z}\right) - \alpha\left(0 + w\frac{\partial p}{\partial z}\right) = 0$$

We can cancel the vertical velocity to get,

$$c_p \frac{\partial T}{\partial z} = \alpha \frac{\partial p}{\partial z} \, .$$

If the pressure is hydrostatic $\partial p/\partial z = -g\rho$, so that,

$$\frac{\partial T}{\partial z} = \frac{1}{c_p}\alpha(-g\rho) = -\frac{g}{c_p} = -\frac{9.8\,\mathrm{m\,s^{-2}}}{1005\ \mathrm{J\ kg^{-1}\ K^{-1}}} = -\frac{9.8\,\mathrm{m\,s^{-2}}}{1005\,\mathrm{m^2 s^2\ K^{-1}}} = -0.00975\,\mathrm{K\ m^{-1}} \, ,$$

or ≈ −9.8 °C km^{-1}. This is the ***Dry Adiabatic Lapse Rate***. It reflects the cooling due to work done by a rising parcel as it expands against the pressure of its surroundings, or conversely, the heating due to work done on a sinking parcel.

If we write the dry adiabatic lapse rate as $-(\partial T/\partial z)_{\mathrm{adiabatic}} = \Gamma_d$ and the existing lapse rate as $-\partial T/\partial z = \Gamma$,

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \frac{\alpha}{c_p}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) - w(\Gamma - \Gamma_d) = \frac{J}{c_p}$$

If $\Gamma < \Gamma_d$, rising air parcel will be cooler than its surroundings, so that it will sink back to its original altitude. If $\Gamma = \Gamma_d$, it will have the same temperature as its surroundings so that the parcel will remain at its displaced altitude. If $\Gamma > \Gamma_d$, the displaced parcel will be warmer than its surroundings and will continue to rise. The process works the same way if the displacement is downward.

- The first case (**$\Gamma < \Gamma_d$**) is said to be **Stable**
- The second case (**$\Gamma = \Gamma_d$**) is said to be **Neutral**
- The third case (**$\Gamma > \Gamma_d$**) is said to be **Unstable**

In the stable case, air parcels undergo vertical ***Buoyancy*** or ***Brunt-Väisälä*** oscillations that have a period of about ten minutes.

Still another way to look at the Thermodynamic Energy Equation is,

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}-\frac{R_dT}{c_pp}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)=\frac{J}{c_p}$$

Dividing through by T and rearranging,

$$\frac{1}{T}\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)-\frac{1}{p}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)\frac{R_d}{c_p}=\frac{J}{c_pT},$$

Recognizing the standard differential forms for ln *T* and ln *p*,

$$\frac{\partial \ln T}{\partial t}+u\frac{\partial \ln T}{\partial x}+v\frac{\partial \ln T}{\partial y}+w\frac{\partial \ln T}{\partial z}-\left(\frac{\partial \ln p}{\partial t}+u\frac{\partial \ln p}{\partial x}+v\frac{\partial \ln p}{\partial y}+w\frac{\partial \ln p}{\partial z}\right)\frac{R_d}{c_p}=\frac{J}{c_pT}$$

Or, rearranging

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\left(\ln T-\frac{R_d}{c_p}\ln p\right)=\frac{D[\ln(Tp^{-R_d/c_p})]}{Dt}=\frac{J}{c_pT}$$

For adiabatic motions, $J = 0$ so that,

$$\frac{D[\ln(Tp^{-R_d/c_p})]}{Dt}=0\Rightarrow Tp^{-R_d/c_p}=\text{constant}$$

If at some pressure, say $p_0 = 1000$ hPa, the temperature is θ,

$$Tp^{-R_d/c_p}=\theta p_0^{-R_d/c_p}\text{, or }\theta=T\left(\frac{p_0}{p}\right)^{R_d/c_p}$$

The quantity θ is called the Potential Temperature. It is the temperature that the air would have if it were brought dry adiabatically to $p_0 = 1000$ hPa. It is ***Conserved***, that is it remains constant, in adiabatic motions with no condensation. Using this definition, we can recognize that Dθ/Dt = D(T/p^{Rdcp})/D*t* + D(p_0^{Rdcp})/D*t* = D(T/p^{Rdcp})Dt , and rewrite the Thermodynamic Equation in still another form,

$$c_p\frac{D(\ln\theta)}{Dt}=\frac{J}{T}=\frac{Ds}{Dt}$$

Here the quantity *s* is ***Entropy*** per unit mass. It is a measure of the unavailability of heat energy. If heat is transferred at a high temperature (lower entropy) more of it is available to do work than if it is transferred at a lower temperature. In an ***Isentropic*** process, no heat is transferred, so that adiabatic processes are also isentropic.

For completeness, let's look again at the Thermodynamic Equation written in terms of lapse rate:

$$\frac{1}{T}\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-\frac{1}{p}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)\frac{R_d}{c_p}-w\frac{(\Gamma-\Gamma_d)}{T}=\frac{J}{c_pT}$$

$$\frac{1}{\theta}\left(\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}\right)+w\frac{(\Gamma_d-\Gamma)}{T}=\frac{J}{c_pT}$$

$$\frac{T}{\theta}\left(\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}\right)=-w(\Gamma_d-\Gamma)+\frac{J}{c_p}$$

If w is downward, ($w < 0$) and $\Gamma < \Gamma_d$, then θ increases with time. If it is upward θ decreases. Thus, even though temperature decreases with height potential temperature increases. In a stable atmosphere the pressure work term dominates the lapse rate.

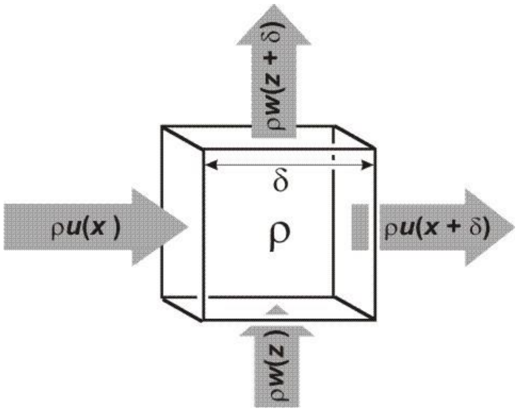
Mass Continuity: Consider a fixed (Eulerian) unit-volume box. If more air flows into than out of the box, the mass inside increases, and conversely. The net flow is the sum of the differences between the flows through opposite faces:

$$\frac{\partial \rho}{\partial t}=-\frac{\partial(\rho u)}{\partial x}-\frac{\partial(\rho v)}{\partial y}-\frac{\partial(\rho w)}{\partial z},$$

$$\frac{\partial \rho}{\partial t}=-u\frac{\partial \rho}{\partial x}-v\frac{\partial \rho}{\partial y}-\frac{\partial \rho}{\partial z}-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}\right),$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+\frac{\partial \rho}{\partial z}=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}\right),$$

$$\frac{D\rho}{Dt}=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}\right)$$



So the Atmospheric form of the complete Navier-Stokes equations is:

$$\frac{Du}{Dt}=fv-\frac{1}{\rho}\frac{\partial p}{\partial x}+\text{Friction}$$

$$\frac{Dv}{Dt}=-fu-\frac{1}{\rho}\frac{\partial p}{\partial y}+\text{Friction}$$

$$\frac{\partial p}{\partial x}=-g\rho$$

$$p_d\alpha=R_dT$$

$$c_p\frac{D\theta}{Dt}=\frac{J}{c_pT}$$

$$\frac{D\rho}{Dt}=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}\right)$$

These are two horizontal momentum equations, the hydrostatic law, equation of state for dry air, thermodynamic energy equation, and mass continuity equation. There are also state and continuity equations for water vapor and various kinds of solid or liquid H_2O. All but the (diagnostic) hydrostatic and state equations are prognostic equations in IVF.