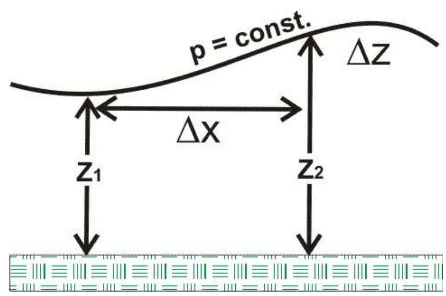
Objective: To cast the equations of motion in pressure coordinates and apply them to simple steady flows.

Reading: CH 2, 46-49; Ch 3, 67-89;

Problems: 3.1, 3.3, 3.5, 3. 9, and 3. 16



Use of pressure instead of height as the vertical coordinate. If one starts on a surface of constant pressure, moves horizontally a distance Δx and then vertically Δz to return to the p surface at a different horizontal position:

$$p+\Delta x\left(\frac{\partial p}{\partial x}\right)_z+\Delta z\left(\frac{\partial p}{\partial z}\right)_x=p\,.$$

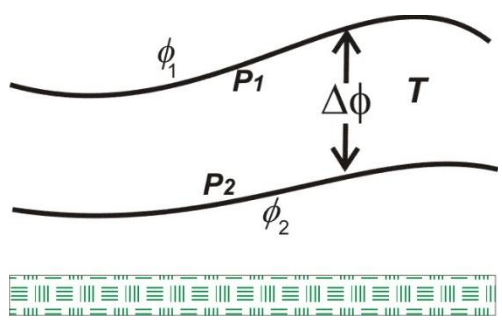
Substituting from the hydrostatic equation and canceling the *p*'s.

$$\Delta x\left(\frac{\partial p}{\partial x}\right)_z+\Delta z\left(-g\rho\right)=0\,,$$

Or

$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z=g\frac{\Delta z}{\Delta x}=g\left(\frac{\partial z}{\partial x}\right)_p$. Similarly for the y direction: $\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z=g\left(\frac{\partial z}{\partial y}\right)_p$. Finally we can write $gz_p = \phi$, the geopotential. With these changes the frictionless, horizontal momentum equations become:

$$\frac{Du}{Dt}-fv=-\frac{\partial\phi}{\partial x}$$
$$\frac{Dv}{Dt}+fu=-\frac{\partial\phi}{\partial y}$$



What does the hydrostatic equation look like in pressure coordinates?

$$\frac{\partial p}{\partial z}=-g\rho=-\frac{gp}{R_dT}=-\frac{p}{H}\,,$$

Where *H* is the scale height. If we turn the equation upside-down and multiply through by *g,*

$$g\frac{\partial z}{\partial p}=\frac{\partial\phi}{\partial p}=-\frac{1}{\rho}=-\alpha=\frac{R_dT}{p}$$

If we use the vertically averaged temperature and integrate upward from level 2 to level 1

$$\int_{p_2}^{p_1}d\phi=\phi_1-\phi_2=\Delta\phi=-R_d\overline{T}\int_{p_2}^{p_1}\frac{dp}{p}=R_d\overline{T}\ln\frac{p_2}{p_1}\,.$$

where $\overline{T}(x,y)$ is the local vertically averaged (virtual) temperature between levels 2 and 1. Since the pressure levels are specified, the only factor that determines the geopotential thickness between the (specified) levels is the average absolute temperature of the layer. Note that $R_d T = gR_d T/g = gH$, which has dimensions of velocity squared m^2 s^{-2}. The above relation is called the ***Hypsometric Equation***.

How does the Thermodynamic Energy Equation look in pressure coordinates?

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = c_p \frac{DT}{Dt} - \frac{R_d T}{p}\frac{Dp}{Dt} = J\ .$$

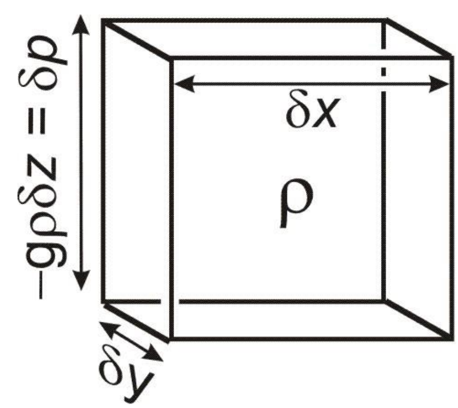
If we define $\omega \equiv Dp/Dt$ and expand the Lagrangian derivative,

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p} - \frac{\alpha}{c_p}\omega = \frac{J}{c_p}\text{, or}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \omega\left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p}\right) = \frac{J}{c_p}\text{, or}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \left(\frac{DT}{Dt}\right)_p - S_p\omega = \frac{J}{c_p},$$ where $S_p \equiv \frac{\alpha}{c_p} - \frac{\partial T}{\partial p}$ is the pressure-coordinate static stability and the subscript *p* on the Lagrangian derivative says that it includes only the *x* and *y* (horizontal on a *p* surface) advections. Note, there is some mathematical sleight-of-hand in going from $w\partial T/\partial z$ to $\omega\partial T/\partial p$ and from horizontal derivatives on *z* surfaces to those on *p* surfaces. Keep in mind also that negative ω represents rising motion and that for a normal lapse rate, $\partial T/\partial p > 0$ because temperature normally increases downward, toward higher pressure.

Continuity in Pressure Coordinates: Here is a Largangian derivation of mass continuity in pressure coordinates. It is complementary with the Eulerian version we did earlier. Consider the Lagrangian derivative of an element of mass $\delta M = \rho\,(\delta x\,\delta y\,\partial z) = (\delta x\,\delta y\,g\rho\partial z)/g = -\,(\delta x\,\delta y\,\partial p)/g$.



$$\begin{aligned}\frac{1}{\delta M}\frac{D\delta M}{Dt} &= \frac{1}{\delta x\delta y\delta p}\frac{D(\delta x\delta y\delta p)}{Dt}\\ &= \frac{1}{\delta x}\frac{D\delta x}{Dt} + \frac{1}{\delta y}\frac{D\delta y}{Dt} + \frac{1}{\delta p}\frac{D\delta p}{Dt}\\ &= \frac{1}{\delta x}\delta\left(\frac{Dx}{Dt}\right) + \frac{1}{\delta y}\delta\left(\frac{Dy}{Dt}\right) + \frac{1}{\delta p}\delta\left(\frac{Dp}{Dt}\right)\\ &= \left(\frac{\partial u}{\partial x}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p}\end{aligned}$$

The pressure-coordinate continuity equation looks like the height-coordinate version without the $D\rho/Dt$ term. Note that there are no approximations involving slow spatial or temporal changes of density here.

Recall that the operator $\partial()/\partial x + \partial()/\partial y + \partial()/\partial z = \nabla\bullet()$ is called the **Divergence**. We will often use the two dimensional version in height coordinates $\partial()/\partial x + \partial()/\partial y = \nabla_h\bullet()$ or in pressure coordinates $(\partial()/\partial x)_p + (\partial()/\partial y)_p = \nabla_p\bullet()$.

The momentum, hydrostatic and thermodynamic equations in pressure coordinates are:

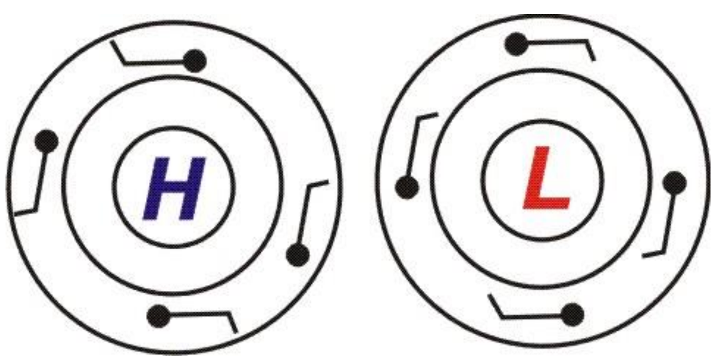
$$\frac{Du}{Dt} - fv = -\frac{\partial \phi}{\partial x}$$
$$\frac{Dv}{Dt} + fu = -\frac{\partial \phi}{\partial y}$$
$$\frac{\partial \phi}{\partial p} = -\alpha = \frac{R_d T}{p}$$
$$\left(\frac{DT}{Dt}\right)_p - S_p \omega = \frac{J}{c_p}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Remember how we derived the hydrostatic equation by showing that the time derivative was much less than gravity. A similar argument can be applied to the time derivative in the Momentum Equations, but it works only in Middle Latitudes, where $f \simeq 10^{-4}$.

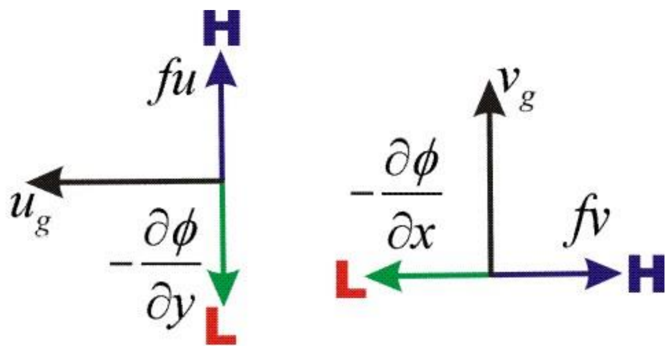
It typically takes a day (~10^5 s) for the wind to change appreciably, and a typical wind speed is about 10 m s^{-1}. Thus, D(*u*,*v*)/Dt ~ (10 m s^{-1}/10^5 s)~ 10^{-4} m s^{-2}, while the Coriolis term ~ *f*(*u*,*v*) ~(10^{-4} s^{-1} x 10 m s^{-1}) ~ 10^{-3} m s^{-2}. This simple scaling argument shows that the time derivative term is about 1/10 of the Coriolis term. Since the geopotential gradient term is the only other term left in the equation, it must equal the Coriolis term. We can therefore write:

$$v_g = \frac{1}{f}\frac{\partial \phi}{\partial x}, \quad \text{or in height coordinates, } v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$
$$u_g = -\frac{1}{f}\frac{\partial \phi}{\partial y}, \quad \text{or in height coordinates, } u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

This argument explains why the wind blows <u>around</u> pressure systems on a middle-latitude weather map, instead of radially in and out.



This balanced flow is termed Geostrophic. It it, the flow is perpendicular to the pressure gradient with the Coriolis force (northern hemisphere) acting to the right of the motion.

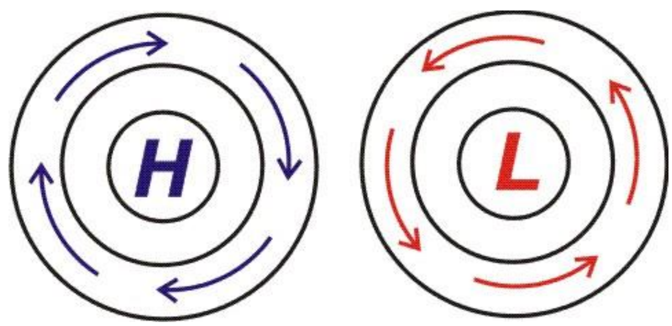


The relation provides a way to compute the wind speed and direction from the distribution of pressure. For example suppose that the north-south pressure gradient is 102 m over a distance of 500 km at 45° N latitude with higher pressure to the north.

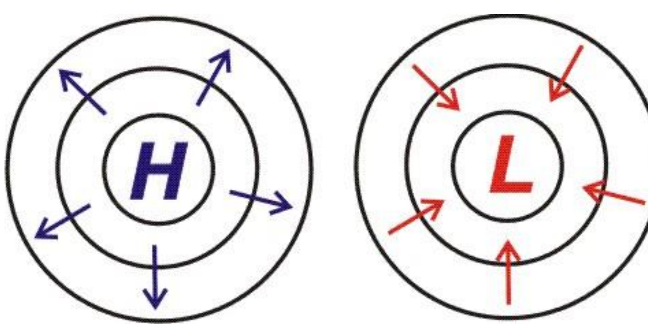
$$u = \frac{(9.8\text{ m s}^{-2})\bullet(102\text{ m})}{(10^{-4}\text{ s}^{-1})\bullet(500\text{ km})\bullet(1000\text{ m/km})} = 19.99\text{ m s}^{-1}$$

Wind will blow from the East, as in the above diagram.

The geostrophic wind suggest that we might want to represent the vector wind in terms of the derivatives of a scalar quantity, ψ, the ***Streamfunction**,* that acts like geopotential divided by the Coriolis parameter. The relationship is $u = -\partial\psi/\partial y, v = \partial\psi/\partial x$. For geostrophic flow, $\psi = gz_p / f_0$. Where f_0 is a constant representative value of *f*.



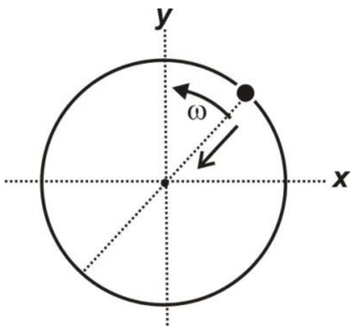
On the other hand, we may want to use a scalar to represent a flow that does not circulate around centers of action, but flows radially in or out of them. For this representation we employ a ***Velocity Potential***, χ , such that $u = -\partial\chi/\partial x, v = -\partial\chi/\partial y$. Any wind field can be represented by a combination of a velocity potential and a streamfunction. A ***Streamline*** is a line of constant streamfunction. It is not the same as the ***Trajectory*** of an air parcel unless the flow is steady over time.



The wind is almost never exactly geostrophic, and it turns out that the ***Ageostrophic Wind*** is the difference between the geostrophic and actual wind (u_a, v_a) = ($u - u_g$, $v - v_g$). Even when the wind is not geostrophic, we can calculate what the geostrophic wind would be, and we can use it to represent the pressure or geopotential gradient, $\partial\phi/\partial x = fv_g, \partial\phi/\partial y = -fu_g$. Since the wind is generally nearly in geostrophic balance, $u_a \ll u$ and $v_a \ll v$. If we substitute into the horizontal momentum equations:

$$\frac{Du}{Dt} - fv + \frac{\partial\phi}{\partial x} = \frac{Du}{Dt} - fv + fv_g = \frac{D(u_g + u_a)}{Dt} - f(v - v_g) \cong \frac{Du_g}{Dt} - fv_a = 0$$
$$\frac{Dv}{Dt} + fu + \frac{\partial\phi}{\partial y} = \frac{Dv}{Dt} + fu - fu_g = \frac{D(v_g + v_a)}{Dt} + f(u - u_g) \cong \frac{Dv_g}{Dt} + fu_a = 0$$

This relation shows that the acceleration of the actual wind---or of the geostrophic wind---is proportional to the ageostrophic wind which is about 1/10 of the total wind. A more formally derived and exact version of this relation is the key to understanding fronts and baroclinic instability.

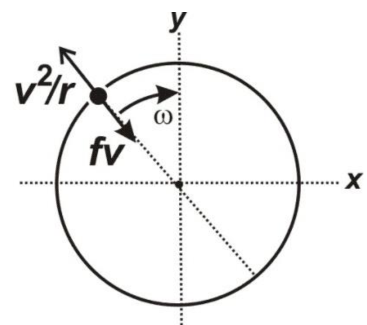


Recall our analysis of the acceleration of the wind when it follows a circular path. Let $\vec{R} = \hat{i}x + \hat{j}y$ be the vector distance from the center of rotation.

$$\vec{R}(t) = R\mathbf{r}$$
$$\frac{DR}{Dt} = R\omega\mathbf{k}\times\mathbf{r} = V\mathbf{k}\times\mathbf{r}$$
$$\frac{D^2R}{Dt^2} = R^2\omega^2\mathbf{k}\times\mathbf{k}\times\mathbf{r} = -R^2\omega^2\mathbf{r} = -\frac{V^2}{R}\mathbf{r}$$

This is the **Cyclostrophic** or ***Centrepetal*** acceleration toward the center of rotation that must be balanced by some force for circular motion to exist.

One possibility is the ***Inertia Circle***, in which the Coriolis and centripetal accelerations balance as the air orbits anticylonically.



$$\frac{V^2}{R} = fV \Rightarrow V = fR$$

The air makes a complete orbit in time

$$\frac{2\pi R}{V} = \frac{2\pi R}{fR} = \frac{2\pi}{f},$$

Which is called a ***Pendulum Day***. The period varies from 35 h at 25° lat, to 24 h at 30°, to 16.4 h at 45°, to 11.96 h at the pole. It is ½ a sidereal (with respect to the stars, not the sun) day divided by the sign of the latitude.

Another simple balanced situation is the ***Cyclostrophic Wind*** where the pressure gradient and centripetal accelerations balance. It happens when V2/R >> fV, that is, when the wind is very strong or the spatial scale is small. Examples of this kind of flow are bathtub drains, dust devils, tornadoes, and hurricanes, at least near the eye.

$$\frac{V_c^2}{R} = \frac{\partial\phi}{\partial R} \Rightarrow V_c = \pm\sqrt{R\frac{\partial\phi}{\partial R}}$$

The cyclostrophic wind can exist only in low pressure systems, but it can be either cyclonic or anticyclonic.

The Gradient Wind involves balance among the geopotential gradient, Coriolis, and cyclostrophic terms.

$$\frac{v_G^2}{r} + fv_G + \frac{\partial\phi}{\partial r} = \frac{v_G^2}{r} + f(v_G - v_g) = 0$$

Re arranging and solving with the quadratic formula:

$$v_G^2 + frv_G - frv_g = 0$$

$$v_G = \frac{-fr \pm \sqrt{f^2r^2 + frv_g}}{2} = \frac{fr}{2}\left(-1 \pm \sqrt{1 + \frac{4v_g}{fr}}\right)$$

Choose the positive root, but multiply and divide by the negative root inside the parentheses.

$$v_G = \frac{fr}{2}\left[\frac{\left(-1+\sqrt{1+\frac{4v_g}{fr}}\right)\left(-1-\sqrt{1+\frac{4v_g}{fr}}\right)}{-1-\sqrt{1+\frac{4v_g}{fr}}}\right] = \frac{fr}{2}\left[\frac{1-\left(1+\frac{4v_g}{fr}\right)}{-1-\sqrt{1+\frac{4v_g}{fr}}}\right] = \frac{fr}{2}\left[\frac{-\frac{4v_g}{fr}}{-1-\sqrt{1+\frac{4v_g}{fr}}}\right]$$

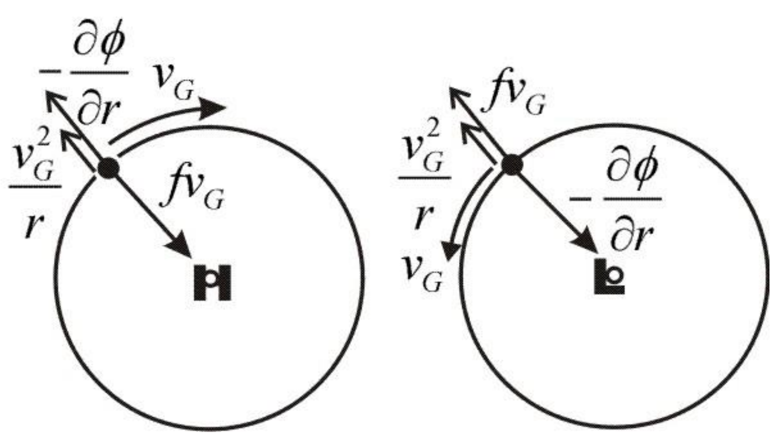
$$v_G = \frac{2v_g}{1+\sqrt{1+\frac{4v_g}{fr}}} = \frac{v_g}{\frac{1}{2}\left[1+\sqrt{1+\frac{4v_g}{fr}}\right]}$$

We define the quantity $v_g / fr = (v_g^2 / r) / fv = \mathrm{R}$ as the Geostrophic Rossby Number. It is the ratio of the cyclostrophic and Coriolis accelerations computed using the geostrophic wind. The final form of the gradient wind equation is

$v_G = \dfrac{v_g}{\frac{1}{2}\left[1+\sqrt{1+\mathrm{R}}\right]}$. When R is small the wind is geostrophic. When R is large,

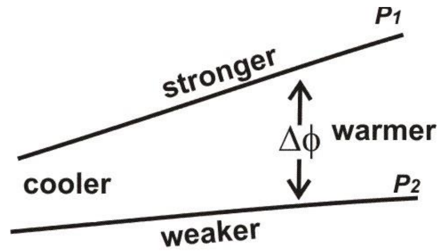
$$v_G = \frac{v_g}{\frac{1}{2}\sqrt{\frac{4v_g}{fr}}} = \frac{v_g}{\sqrt{\frac{v_g}{fr}}} = \sqrt{frv_g} = \sqrt{rfv_g} = \sqrt{r\frac{\partial \phi}{\partial r}} = V_c$$

The balance of forces in gradient flow for high and low pressure systems:



In anticyclones the geopotential gradient and cyclostrophic accelerations add and together balance the Coriolis acceleration. In cyclones the cyclostrophic and geostrophic accelerations together balance the pressure gradient. Consequently, winds in an anticyclone will be stronger than in a cyclone for the same pressure gradient. The circulations go the other way in the southern hemisphere, but the balance of forces works nearly the same way, but it is a mirror image. A rare phenomenon, called ***Anomalous Flow*** arises from selection of the negative root in the quadratic solution.

The ***Thermal Wind*** is the vertical change of the geostrophic wind between two pressure levels caused by gradient of temperature in the layer bounded by the pressures. It combines the hypsometeric equation with the geostrophic wind relations in each level.

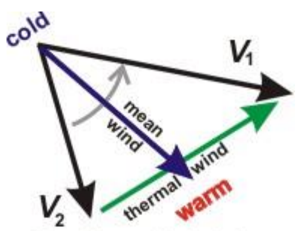


$$\phi_1 - \phi_2 = R_d \bar{T} \ln\frac{p_2}{p_1}$$

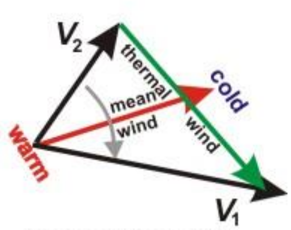
$$v_1 = \frac{1}{f}\frac{\partial \phi_1}{\partial x}, \quad v_2 = \frac{1}{f}\frac{\partial \phi_2}{\partial x}, \quad u_1 = -\frac{1}{f}\frac{\partial \phi_1}{\partial y}, \quad u_2 = -\frac{1}{f}\frac{\partial \phi_2}{\partial y}.$$

$$v_1 - v_2 = \frac{1}{f}\left(\frac{\partial \phi_1}{\partial x} - \frac{\partial \phi_2}{\partial x}\right) = \frac{R_d}{f}\frac{\partial \bar{T}}{\partial x}\ln\frac{p_2}{p_1},$$

$$u_1 - u_2 = -\frac{1}{f}\left(\frac{\partial \phi_1}{\partial y} - \frac{\partial \phi_2}{\partial y}\right) = -\frac{R_d}{f}\frac{\partial \bar{T}}{\partial y}\ln\frac{p_2}{p_1}$$



Backing with height



Veerng with height

The pressures are constant because they define the layer over which the shear occurs. Only the temperature gradient affects the shear. The thermal wind is the vertical difference of the geostrophic wind across a layer defined by two pressure surfaces. It has the same relation to gradients of geopotential thickness that geostrophic wind has to gradients of geopotential. Low thickness between isobaric surfaces corresponds to cold temperatures. As with pressure and the geostrophic wind, if an observer stands with the thermal wind at her back, low thickness (cold air) lies to her left. Thus, cold advection occurs when the thermal wind backs (turns counter clockwise in the northern hemisphere) with height, and cold advection occurs when it veers (turns clockwise) with height.

Another look at the Continuity Equation in Height Coordinates:

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) = -\left[u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = \left(\frac{D\rho}{Dt}\right)_h = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{w}{\rho}\frac{\partial \rho}{\partial z}\right)$$

Recall that from the hydrostatic equation pressure fluctuations result from changes in the integrated density aloft. The surface pressure is about 1000 hPa and it typically fluctuates by about 10 hPa. From the hydrostatic law , ρ'/ ρ, the ratio of day-to-day variations of density as synoptic systems pass to the typical density, is approximately p'/p = 0.01, the ratio of pressure fluctuations to the total pressure. Since we are dealing with synoptic scale systems, the timescale for these changes is about 10^5 s. Thus, $1/\rho\ \partial\rho/\partial t = 0.01 \times 10^{-5} \sim 10^{-7}\ s^{-1}$.

On the other hand, the divergence terms scale as a typical velocity 10 m s^{-1} divided by 1000 km = 10^6 m ~ 10^{-5} s^{-1}. This argument overestimates the divergence because the real divergence is actually the divergence of the ageostrophic wind, which is an order of magnitude weaker than the total wind. Even with this more sophisticated estimate, the divergence is still an order of magnitude greater than the Largangian derivative of the density divided by the density itself. With this approximation, and substituting from the hydrostatic equation as indicated above:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}-\frac{w}{H}=0$$

Note that this analysis eliminates the time derivatives of density (as in pressure coordinates) and reduces continuity to a diagnostic relation. Furthermore, the vertical scale of the middle-latitude atmosphere is only a bit larger than the scale height. If the horizontal velocity and length scales are U ~ 10 m s^{-1} and L ~1000 km = 10^6 m, and the vertical velocity and length scales are W and H ~7000 m, W ~ UH/L = (10 m s^{-1})(7000 m) /(10^6 m) ~ 0.07 m s^{-1} = 7 cm s^{-1}. If we replace the total wind here with the ageostrophic wind W becomes even smaller, ~1 cm s^{-1}. In later arguments we will often also neglect the w/H term in the continuity equation, even though the justification is weak.