Objective: To introduce the dynamics and kinematics of vorticity and divergence

Reading: CH 4, pp 95-110;

Problems: 4.1, 4.2, 4.3, 4.5 & 4.11 – pp 122 & 113

Recall how we simplified the Continuity Equation <u>in Height Coordinates</u>:

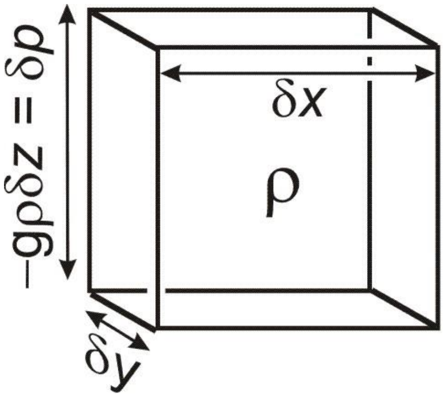
$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}=\left(\frac{D_H\rho}{Dt}\right)=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\frac{w}{\rho}\frac{\partial\rho}{\partial z}\right)$$

From the hydrostatic equation, $\partial\rho/\partial z=-\rho/H$, where H is the scale height. Since fluctuations of the surface pressure are ≈1% of the pressure itself, the hydrostatic law implies that we can neglect ($D_H\rho/Dt$), and write the divergence in height coordinates as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}-\frac{w}{H}=0$$

This result eliminates the time derivatives of density and reduces mass conservation to a diagnostic relation. Often in later arguments we will also neglect the *w*/*H* term, even though there is limited justification.

Continuity in Pressure Coordinates: The Largangian derivation of mass continuity in pressure coordinates is complementary with the Eulerian version in height coordinates. As discussed previously, we follow an element of mass δ*M* = ρ (δx δy ∂z) = (δx δy gρ∂z)/g = − (δx δy ∂p)/g.



$$\frac{1}{\delta M}\frac{D\delta M}{Dt}=\left(\frac{\partial u}{\partial x}\right)_p+\left(\frac{\partial v}{\partial y}\right)_p+\frac{\partial\omega}{\partial p}=0$$

The pressure-coordinate continuity equation looks like the simplified height-coordinate version without the last term that contains the scale height. Note that there are no approximations involving slow spatial or temporal changes of density here.

The operator $\partial(\,)/\partial x+\partial(\,)/\partial y+\partial(\,)/\partial z=\nabla\bullet(\,)$ is the **Divergence**. We often use the two dimensional version in height coordinates $\partial(\,)/\partial x+\partial(\,)/\partial y=\nabla_h\bullet(\,)$, or in pressure coordinates $(\partial(\,)/\partial x)_p+(\partial(\,)/\partial y)_p=\nabla_p\bullet(\,)$.

Vorticity equation in pressure coordinates: Start with the momentum equations in pressure coordinates:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial p}-fv=-\frac{\partial\phi}{\partial x}$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial v}{\partial p}+fu=-\frac{\partial\phi}{\partial y}$$

We want to eliminate the geopotential from this pair of equations by taking −∂/∂y of the top and ∂/∂x of the bottom:

$$\frac{\partial}{\partial t}\left(-\frac{\partial u}{\partial y}\right)+\left(-\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x}+u\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)+\frac{\partial v}{\partial y}\left(-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)-\frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}+\omega\frac{\partial}{\partial p}\left(-\frac{\partial u}{\partial y}\right)$$
$$+f\frac{\partial v}{\partial y}+\frac{\partial f}{\partial y}v=\frac{\partial^2\phi}{\partial x\partial y}$$
$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\left(\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial y}+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p}+\omega\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial x}\right)$$
$$+f\frac{\partial u}{\partial x}=-\frac{\partial^2\phi}{\partial x\partial y}$$

Rearranging some terms

$$\frac{\partial}{\partial t}\left(-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)+\omega\frac{\partial}{\partial p}\left(-\frac{\partial u}{\partial y}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(-\frac{\partial u}{\partial y}\right)-\frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}$$
$$+f\frac{\partial v}{\partial y}+\frac{\partial f}{\partial y}v=\frac{\partial^2\phi}{\partial x\partial y}$$
$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)+\omega\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial x}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x}+\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p}+f\frac{\partial u}{\partial x}=-\frac{\partial^2\phi}{\partial x\partial y}$$

Adding:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\omega\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$+\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)f+\frac{\partial f}{\partial y}v=0$$

Simplifying:

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+\omega\frac{\partial}{\partial p}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+v\frac{\partial f}{\partial y}+\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}=0$$

Since *f* is not a function of *x*, *t*, or *p*:

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+\omega\frac{\partial}{\partial p}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}=0$$

Finally:

$$\left(\frac{D_H}{Dt}+\omega\frac{\partial}{\partial p}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)-\frac{\partial \omega}{\partial p}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}=0$$

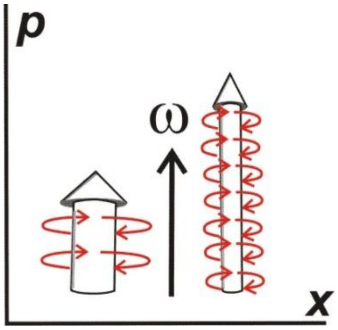
Or if $\zeta = \partial v/\partial x - \partial u/\partial y$ is defined to be the relative vorticity.

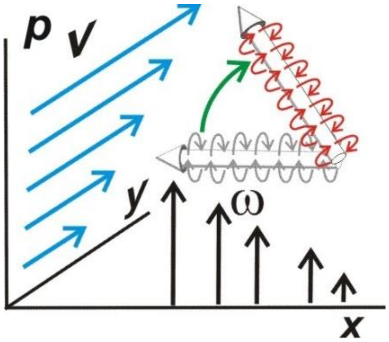
$$\frac{D_H(\zeta+f)}{Dt}+\omega\frac{\partial(\zeta+f)}{\partial p}-\frac{\partial \omega}{\partial p}(\zeta+f)+\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}=0$$

The first term is the “horizontal” Lagrangian derivative of the vertical component of absolute vorticity, which is composed of the relative vorticity ζ and the planetary vorticity f. Although the vorticity looks like a scalar, what we have derived here is actually the vertical component of a vector (see below) that also has horizontal components that stem from the vertical shear of the horizontal wind ($\partial u/\partial p$, $\partial v/\partial p$) and horizontal shear of the vertical wind ($\partial \omega/\partial x$, $\partial \omega/\partial y$). The directions of vorticity vectors are determined using the right-hand rule. When your right hand encircles the vorticity vector with the fingers wrapping around it in the direction of rotation, the thumb points in the same direction as the vorticity.

The second term is the vertical advection of the vorticity.

The third term is the vorticity stretching term. Convergence concentrates the vortex tubes, reducing the area of rotating fluid and increasing the rate of rotation. Divergence disperses the vortex tubes, increasing the area and slowing the rate of rotation. This mechanism is important to all scales of atmospheric motions, including frontal cyclones.





The last two terms compose the tilting term, which tips horizontal vorticity into the vertical. They are often neglected in synoptic meteorology, but they are important in mesoscale phenomena like tornadoes. In the case shown, the horizontal shear due meridional wind that increases upward is being tilted into the vertical by ω that increases in the positive x direction by becoming less negative---not so strong upward toward low pressure.

The three dimensional vorticity in height coordinates is:

$$\nabla\times\vec{V}=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\hat{i}-\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)\hat{j}+\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial y}\right)\hat{k}$$

The $\nabla\times$ operator is the vector **Curl**. If we define $\beta \equiv \partial f/\partial y$, another way to write the vorticity equation is,

$$\frac{D\zeta}{Dt}+\omega\frac{\partial\zeta}{\partial p}+\beta v-\frac{\partial\omega}{\partial p}(\zeta+f)+\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}=0$$

A more elegant way to derive the vorticity equation transforms the momentum equations as follows:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial p}-fv=-\frac{\partial\phi}{\partial x}\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial v}{\partial p}+fu=-\frac{\partial\phi}{\partial y}
\end{aligned}$$

Adding and subtracting $v\partial v/\partial x = \partial(\tfrac{1}{2}v^2)/\partial x$ in the first and $u\partial u/\partial y = \partial(\tfrac{1}{2}u^2)/\partial y$

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\frac{u^2}{2}+\frac{\partial}{\partial x}\frac{v^2}{2}-v\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial p}-fv=-\frac{\partial\phi}{\partial x}\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}-u\frac{\partial u}{\partial y}+\frac{\partial}{\partial y}\frac{u^2}{2}+\frac{\partial}{\partial y}\frac{v^2}{2}+\omega\frac{\partial v}{\partial p}+fu=-\frac{\partial\phi}{\partial y}
\end{aligned}$$

Combining some terms

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}\right)-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\omega\frac{\partial u}{\partial p}-fv=-\frac{\partial\phi}{\partial x}\\
&\frac{\partial v}{\partial t}+u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{u^2}{2}+\frac{v^2}{2}\right)+\omega\frac{\partial v}{\partial p}+fu=-\frac{\partial\phi}{\partial y}
\end{aligned}$$

And collecting them

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\omega\frac{\partial u}{\partial p}-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=-\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\phi\right)\\
&\frac{\partial v}{\partial t}+\omega\frac{\partial v}{\partial p}+u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=-\frac{\partial}{\partial y}\left(\frac{u^2}{2}+\frac{v^2}{2}+\phi\right)
\end{aligned}$$

Differentiating as before ($-\partial/\partial y$ of top equation and $\partial/\partial x$ of the bottom):

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(-\frac{\partial u}{\partial y}\right)-\frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}+\omega\frac{\partial}{\partial p}\left(-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=\frac{\partial^2}{\partial x\partial y}\left(\frac{u^2}{2}+\frac{v^2}{2}+\phi\right)\\
&\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p}+\omega\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=-\frac{\partial^2}{\partial y\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\phi\right)
\end{aligned}$$

Adding makes the terms on the right cancel.

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}+\omega\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=0$$

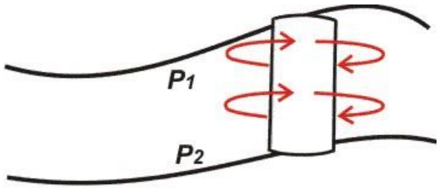
Recognizing that $\partial f/\partial t = 0$, first term on the left becomes the Eulerian derivative of the absolute vorticity. The vertical momentum advection terms on the left become the vertical vorticity advection and tilting terms, and the final terms on the left become isobaric (i.e., “horizontal” in p coordinates) advection and stretching terms. This approach is somewhat simpler and easier to understand if you have the insight to set it up right. Rearranging a bit more,

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\omega\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}=0$$

Identifying the Lagrangian derivative and substituting from the mass continuity equation yields,

$$\frac{D_H}{Dt}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)-\frac{\partial \omega}{\partial p}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}=0$$

Potential vorticity: In Barotropic flow, the temperature gradients are zero so that by the thermal wind equation the vertical shears of the horizontal wind components are zero which, in turn, makes the vertical advection and tilting terms zero so that the vorticity equation simplifies to:



$$\frac{D(\zeta+f)}{Dt}-\frac{\partial \omega}{\partial p}(\zeta+f)=0$$

Imagine two isobaric surfaces at pressures p_1 and p_2. $D(p_2 - p_1)/dt = Dp_2/Dt - Dp_1/Dt = \omega_2 - \omega_1$. We can write $\partial\omega/\partial p = (\omega_2 - \omega_1)/(p_2 - p_1) = (D\Delta p/Dt)/\Delta p$. Substituting into the vorticity equation:

$$\frac{D(\zeta+f)}{Dt}-\frac{1}{\Delta p}\frac{D\Delta p}{Dt}(\zeta+f)=0$$

Dividing by Δp

$$\frac{1}{\Delta p}\frac{D(\zeta+f)}{Dt}-\frac{1}{(\Delta p)^2}\frac{D\Delta p}{Dt}(\zeta+f)=\frac{D}{Dt}\left(\frac{\zeta+f}{\Delta p}\right)=0$$

This quantity is the Potential Vorticity (PV). Some variation of PV is conserved following air parcels in many atmospheric flows— including many that are not barotropic.

Barotropic, nondivergent vorticity equation. Suppose that the flow is also nondivergent, $\partial u/\partial x + \partial v/\partial y = 0$ as well as barotropic. Lets represent the flow using a streamfunction such that $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial y$. The vorticity equation with the planetary vorticity advection explicitly taken outside the Lagarangian derivative becomes:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial f}{\partial y}=0$$

By substitution $\zeta = \partial v/\partial x - \partial u/\partial y = \partial^2\psi/\partial x^2 + \partial\psi^2/\partial y^2$ so that:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial y^2}\right)-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial y^2}\right)+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial y^2}\right)+\frac{\partial f}{\partial y}\frac{\partial\psi}{\partial x}=0$$

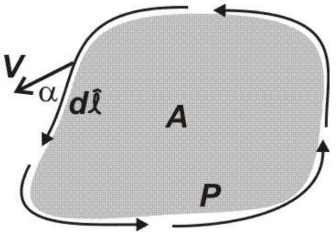
We now make the Middle-Latitude Beta-Plane Approximation where $\beta \equiv \partial f/\partial y = \partial(2\Omega\sin\varphi)/\partial(A\varphi) = 2\Omega\cos\varphi/A = f\cot\varphi/A$ is the meridional gradient of the Coriolis parameter at latitude φ and A is the radius of the Earth such that $y = A\varphi$. All of which transforms the barotropic nondivergent vorticity equation into.

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial y^2}\right)-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial y^2}\right)+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial y^2}\right)+\beta\frac{\partial\psi}{\partial x}=0$$

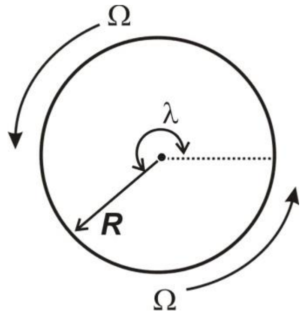
As before, the operator $\nabla_p^2(\,)=\partial^2(\,)/\partial x^2+\partial^2(\,)/\partial y^2$ is the **Laplacian.** Its three-dimensional counterpart in height coordinates is $\nabla_p^2(\,)=\partial^2(\,)/\partial x^2+\partial^2(\,)/\partial y^2+\partial^2(\,)/\partial z^2$.

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2\psi\right)+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2\psi\right)+\beta\frac{\partial\psi}{\partial x}=0$$

Vorticty and circulation: We define the circulation around a closed curve as the integral of the component of the flow locally parallel with the perimeter P integrated around the perimeter. $C=\oint_P \vec{V}\bullet d\hat{\ell}=\oint_P \left|\vec{V}\right|\cos\alpha\, d\hat{\ell}$.



For example consider a disk rotating with angular velocity Ω:



$$C=\oint_{2\pi}(\Omega r)r\,d\lambda=(\Omega r)(2\pi r)=(2\Omega)(\pi r^2)$$

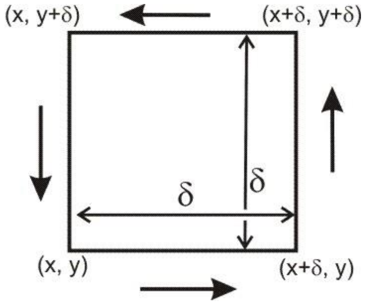
Which is just twice the angular velocity times the area of the disk.

This arrangement can be generalized to any closed curve. Consider the circulation around a square perimeter with sides δ.

$$C = \int_P \vec{V}\bullet d\hat{\ell} = u(y)\delta + v(x+\delta)\delta - u(y+\delta)\delta - v(x)\delta$$

$$= \delta^2[(v(x+\delta)-v(x))/\delta - (u(y+\delta)-u(y))/\delta]$$

$$= A\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial y}\right) = A\zeta$$



Since any area, *A*, bounded by a perimeter, *P*, can be composed of squares like this one, we may write:

$$\oint_P \vec{V}\bullet d\hat{\ell} = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA$$

One can show that the vorticity is, in fact, twice the angular velocity of a rotating object. The above integral relation is called the **Circulation Theorem**. It states that the area integral of the total vorticity encompassed by a closed curve is equal to the line integral of the circulation around that curve.

Vorticity Inversion: Often one has a distribution of vorticity ζ(x, y) and wants to find the corresponding streamfunction ψ(x, y), which obeys Poisson's equation, $\nabla^2\psi = \zeta$ If ζ(*x*,*y*) = Zexp{*i*(*kx* + *ℓy*), we can write *ψ* = Ψ exp{*i*(*kx* + *ℓy*)

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -(k^2+\ell^2)\Psi\exp\{i(kx+\ell y)\} = \mathrm{Z}\exp\{i(kx+\ell y)\}.$$

So that $\Psi = -\mathrm{Z}/(k^2 + \ell^2)$. Thus for a purely sinusoidal vorticity distribution, the streamfunction is the negative of the vorticity scaled by the square of the horizontal wavenumber vector. If the vorticity has are multiple components characterized by different values of k and ℓ, this inversion process deemphasizes the short wavelength (large k or ℓ) so that it acts as a smoother. The inversion process works much the same in three dimensions and also for other meteorological variables that obey Poisson's equation.

Readily applied numerical techniques for solving Poisson's equation also exist.