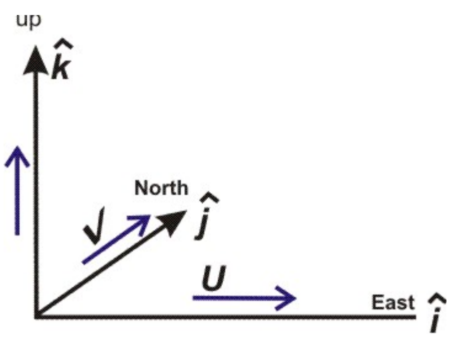
Objective: To introduce the rigorously formulated Quasigeostrophic approximation, including the geostrophic momentum, height-tendency, and omega equations.

Reading: CH 6, 139-162 and 164-168

Problems: Handout

Goal of Dynamic Meteorology: To understand the observed structure and momentum of the large-scale (middle latitude) atmosphere in terms of the fundamental laws of classical physics.



Approximations help in middle latitudes:

- Hydrostatic
- Geostrophic
- Adiabatic
- Beta Plane
- Small divergent component

These combine to produce the formal quasigeostrophic approximation, under which middle-latitude motions seem simpler than tropical or (perhaps) planetary-scale ones.

As we have seen, we can write the frictionless **Horizontal Momentum Equations** in pressure coordinates as:

$$\frac{D\vec{v}}{Dt} + f\hat{k}\times\vec{v} = -\nabla\phi .$$

Where the velocity vector, and the gradient operator apply on a quasi-horizontal isobaric surface.

The **Hydrostatic Equation**, again in pressure coordinates:

$$\frac{\partial\phi}{\partial p} = -\alpha = -\frac{R_d T}{p} .$$

Mass Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial\omega}{\partial p} = \nabla\bullet\vec{v} + \frac{\partial\omega}{\partial p} = 0 .$$

And **Thermodynamic Energy Equation**:

$$c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}\right) - \alpha\omega = J$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+\left(\frac{\partial T}{\partial p}-\frac{\alpha}{c_p}\right)\omega=\frac{J}{c_p}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}-S_p\omega=\frac{J}{c_p}$$

Where:

$$-S_p\equiv\left(-\frac{R_dT}{c_pp}+\frac{\partial T}{\partial p}\right)=\left(\frac{p_0}{p}\right)^{-R/c_p}\left[-\frac{R}{c_p}\frac{T}{p}\left(\frac{p_0}{p}\right)^{R/c_p}+\frac{\partial T}{\partial p}\left(\frac{p_0}{p}\right)^{R/c_p}\right]$$
$$=\frac{T}{T}\left(\frac{p_0}{p}\right)^{-R/c_p}\left[T\frac{\partial}{\partial p}\left(\frac{p_0}{p}\right)^{R/c_p}+\frac{\partial T}{\partial p}\left(\frac{p_0}{p}\right)^{R/c_p}\right]=\frac{T}{T\left(\frac{p_0}{p}\right)^{R/c_p}}\left[\frac{\partial}{\partial p}T\left(\frac{p_0}{p}\right)^{R/c_p}\right]=\frac{T}{\theta}\frac{\partial\theta}{\partial p}$$

Note that since θ increases upward, toward decreasing pressure, $-S_p < 0$. The Thermodynamic Energy Equation becomes:

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}-S_p\omega=\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+\frac{T}{\theta}\frac{\partial\theta}{\partial p}\omega=\frac{J}{c_p}$$
$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+\frac{1}{\left(\frac{p_0}{p}\right)^{R/c_p}}\frac{\partial\theta}{\partial p}\omega=\frac{J}{c_p}$$
$$\left(\frac{p_0}{p}\right)^{R/c_p}\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)+\omega\frac{\partial\theta}{\partial p}=\frac{J}{c_p}\left(\frac{p_0}{p}\right)^{R/c_p}=\frac{\partial\theta}{\partial t}+u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}+\omega\frac{\partial\theta}{\partial p}=\frac{J}{c_p}\left(\frac{p_0}{p}\right)^{R/c_p}$$

Boussinesq Approximation:

Think of the vertical pressure gradient and gravity as they appear in the vertical momentum equation in height coordinates. Here, the perturbation pressure and density p' and ρ' are small compared with the corresponding mean values, p_0 and ρ_0, and the mean pressure is assumed to be hydrostatic, $\rho_0^{-1}\partial p_0/\partial z=-g$.

$$-\frac{\partial p}{\partial z}-g=-\frac{1}{\rho_0+\rho'}\frac{\partial}{\partial z}(p_0+p')-g\approx-\frac{1}{\rho_0}\left(1-\frac{\rho'}{\rho_0}\right)\left(\frac{\partial p_0}{\partial z}+\frac{\partial p'}{\partial z}\right)-g$$
$$=-\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}+\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}\frac{\rho'}{\rho_0}-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}+\frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial z}-g$$
$$=-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}-g\frac{\rho'}{\rho_0}$$

The second term in the last line is the buoyancy. In the Boussinesq approximation one neglects the difference between the actual density and the mean density everywhere except when it is multiplied by g, so that the right side of the vertical momentum equation is written:

$$
\begin{aligned}
-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\rho'}{\rho_0} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\frac{p_0 T'/RT_0^2}{p_0/RT_0} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\frac{T'}{T_0} \\
&= -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\frac{T'(1000\,\text{mb}/p_0)}{T_0(1000\,\text{mb}/p_0)} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\frac{\theta'}{\theta_0}.
\end{aligned}
$$

Note that $\rho = p_0 / R[T_0 + (T - T_0)] = p_0 / RT_0[1 + T'/T_0] \approx (p_0 / RT_0)(1 - T'/T_0)$. This is a hokey derivation, but an exact derivation to be done next term gives the same result. The quantity $g\theta'/\theta_0$ is called the buoyancy, under which the momentum equations take the form:

$$
\begin{aligned}
\frac{Du}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv, \\
\frac{Dv}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu, \\
\frac{Dw}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta'}{\theta_0}, \\
\frac{\partial u}{\partial x} &+ \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{aligned}
$$

Which the book uses in its derivations. Note that ρ_0 is the same at all levels. This notion works much better in the sea than in the atmosphere. We will work in pressure coordinates.

The quasigeostrophic (QG) wind

Remember the geostrophic wind:

$u_g = -\frac{1}{f}\frac{\partial \phi}{\partial y}, v_g = \frac{1}{f}\frac{\partial \phi}{\partial x}$, or in vector notation, $\mathbf{v}_g = f^{-1}\mathbf{k} \times \nabla \phi$.

Since the geostrophic wind is both an approximation to the real wind and another way to represent the geopotential gradient, we may write:

$$
f\mathbf{k} \times \vec{v}_g = f\mathbf{k} \times (u_g\mathbf{i} + v_g\mathbf{j}) = f(u_g\mathbf{j} - v_g\mathbf{i}) = f\left(-\frac{1}{f}\frac{\partial \phi}{\partial y}\mathbf{j} - \frac{1}{f}\frac{\partial \phi}{\partial x}\mathbf{i}\right) = -\left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j}\right) = -\nabla \phi
$$

So substituting into the vector momentum equation:

$$
\frac{D\mathbf{v}}{Dt} + f\mathbf{k} \times \vec{v} = -\nabla \phi = f\mathbf{k} \times \mathbf{v}_g
$$

Or:

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k}\times(\mathbf{v}-\mathbf{v}_g) = \frac{D\mathbf{v}}{Dt} + f\mathbf{k}\times\mathbf{v}_a = 0\,,$$

Where $\mathbf{v}_a \equiv \mathbf{v}-\mathbf{v}_g$. We nondimensionalize, representing each variable, q = Q{q}, as a scale factor (Q, a typical "maximum" value) a nondimensional counterpart {q} of the variable {in curly brackets} which takes on a value ≈1. The nondimensional variables used here are: $(u, v, u_g, v_g) = V\{u, v, u_g, v_g\}$, $(x, y) = L\{x, y\}$, $t = (L/V)\{t\}$, where as before, V ≈ 10 m s^{-1}, L ≈ 1000 km = 10^6 m, and T = L/V = 10^6 m/10 m s^{-1} = 10^5 s ≈ a day, or 86400s. Remember that at lat 45° $f = 10^{-4}$.The nondimensional vector momentum equation is:

$$\frac{V^2}{L}\left\{\frac{D\mathbf{v}}{Dt}\right\} + fV\left\{\mathbf{k}\times(\mathbf{v}-\mathbf{v}_g)\right\} = 0$$

Which rearranges to:

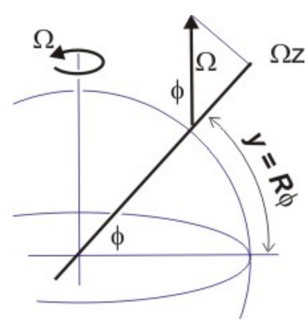
$$\left\{\frac{D\mathbf{v}}{Dt}\right\} + \frac{fL}{V}\left\{\mathbf{k}\times(\mathbf{v}-\mathbf{v}_g)\right\} = \left\{\frac{D\mathbf{v}}{Dt}\right\} + \frac{1}{\mathrm{Ro}}\left\{\mathbf{k}\times(\mathbf{v}-\mathbf{v}_g)\right\} = 0$$

The magnitude of Ro = V/fL = 10/(10^{-4})(10^6) = 0.1. Thus, Ro times the acceleration of the <u>nondimensional</u> total wind is equal to **k** x the nondimensional ageostrophic wind. One way to resolve the apparent contradiction is for the ageostrophic wind $\mathbf{v}_a = \mathbf{v} - \mathbf{v}_g \approx \mathrm{Ro}\,\{\mathbf{v}_g\} \ll \mathbf{v}_g$. If this is true we can write the Lagrangian derivative:

$$\begin{aligned}\frac{D}{Dt} &= \frac{V}{L}\left\{\frac{\partial}{\partial t} + (u_g + u_a\mathrm{Ro})\frac{\partial}{\partial x} + (v_g + v_a\mathrm{Ro})\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}\right\}\\ &\simeq \frac{V}{L}\left\{\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}\right\} \equiv \frac{D_g}{Dt}\end{aligned}$$

Remember when we worked out what the meridional gradient of $f = 2\Omega\sin\varphi$ was.

$$f = f_0 + \frac{\partial f}{\partial y}(y - y_{0)})\,,$$



Where $y = R\varphi$ is the distance measured from the equator, R is the radius of the Earth, and y_o puts us at 45°, in the middle of the middle-latitude band where $f = f_o = 10^{-4}$.

$$\beta \equiv \frac{\partial f}{\partial y} = \frac{\partial}{\partial\varphi}(2\Omega\sin\varphi)\frac{\partial\varphi}{\partial y} = 2\Omega\cos\varphi\frac{\partial}{\partial y}\left(\frac{y}{R}\right) = \frac{1}{R}2\Omega\cos\varphi = \frac{f_0}{R}$$

At 45° latitude, where cos φ = sin φ, we can write $f = f_o(1 + y/A) = f_o(1 + \{y\}L/A)$. The radius of the Earth is 6371 km and L is about 1000 km, so that L/A = 0.16 ≈ 0.1 ≈ Ro. So we can write $f = f_0 + \beta y = f_0\,(1+ \mathrm{Ro}\{\beta y\})$.

Let's take a look at the geopotential gradient, written in terms of the geostrophic wind again.

$$f\mathbf{k}\times\mathbf{v}_g=(f_0+\beta y)\mathbf{k}\times\mathbf{v}_g=f_0(1+\mathrm{Ro}\{\beta y\})\mathbf{k}\times\mathbf{v}_g\simeq f_0\mathbf{k}\times\mathbf{v}_q\equiv-\nabla\phi$$

This is the definition of the **Quasi-Geostrophic** (quasigeostrophic, or QG) wind. It is the geostrophic wind calculated with the Coriolis parameter fixed at 10^{-4}. As with the ordinary geostrophic wind, we use it both as an approximation to the actual wind and as a convenient way to write the pressure gradient. From now on when we write $\mathbf{v}_q$, we mean the QG wind, and when we write $\mathbf{v}_a$ we mean the difference between the QG and actual wind. How does all of this effect the momentum equation that we worked out earlier?

$$\frac{D_g\mathbf{v}_q}{Dt}+(f_0+\beta y)\mathbf{k}\times(\mathbf{v}_q+\mathbf{v}_a)=f_0\mathbf{k}\times\mathbf{v}_q\equiv-\nabla\phi$$

Or in nondimensional form

$$\mathrm{Ro}\left\{\frac{D_g\mathbf{v}_q}{Dt}\right\}+\{\mathbf{k}\times\mathbf{v}_q\}+\mathrm{Ro}\{\mathbf{k}\times\mathbf{v}_a\}+\mathrm{Ro}\{\beta y\mathbf{k}\times\mathbf{v}_q\}+\mathrm{Ro}^2\{\beta y\mathbf{k}\times\mathbf{v}_a\}=\{\mathbf{k}\times\mathbf{v}_q\}$$

Canceling the second term on the left with the only term on the right and dividing through by Ro produces:

$$\left\{\frac{D_g\mathbf{v}_g}{Dt}\right\}+\{\mathbf{k}\times\mathbf{v}_a\}+\{\beta y\mathbf{k}\times\mathbf{v}_g\}+\mathrm{Ro}\{\beta y\mathbf{k}\times\mathbf{v}_a\}=0$$

Since Ro ~ 0.1, the last term on the right may be neglected. Back in dimensional form:

$$\frac{D_g\mathbf{v}_q}{Dt}+f_0\mathbf{k}\times\mathbf{v}_a+\beta y\mathbf{k}\times\mathbf{v}_q=0$$

This is the quasigeostrophic momentum equation. The key trick in deriving it is to recognize that we can neglect advection of planetary vorticity by the ageostrophic wind both because the ageostrohic wind is small compared with the geostrophic (or actual) wind and because the variation of the Coriolis parameter across the domain is small compared with the Coriolis parameter itself.

The definition of the QG wind provides insight into the Continuity Equation. Consider the divergence.

$$\nabla\bullet\mathbf{v}=\nabla\bullet(\mathbf{v}_q+\mathbf{v}_a)=\frac{\partial u_q}{\partial x}+\frac{\partial v_q}{\partial y}+\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}=\left[\frac{\partial}{\partial x}\left(-\frac{1}{f_0}\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{1}{f_0}\frac{\partial\phi}{\partial x}\right)\right]+\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}$$
$$=\left[-\frac{1}{f_0}\frac{\partial^2\phi}{\partial x\partial y}+\frac{1}{f_0}\frac{\partial^2\phi}{\partial x\partial y}\right]+\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}=\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}=\nabla\bullet\mathbf{v}_a$$

We invoke the hydrostatic relation to get nondimensional p = $-\rho gH\{p\}$ leading to ω = $-\rho g\mathrm{W}\{\omega\}$, so that the nondimensional continuity equation becomes:

$$\mathrm{Ro}\frac{V}{L}\left\{\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}\right\}+\frac{\rho g W}{\rho g H}\left\{\frac{\partial \omega}{\partial p}\right\}=\mathrm{Ro}\frac{V}{L}\left\{\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}\right\}+\frac{W}{H}\left\{\frac{\partial \omega}{\partial p}\right\}=0$$

Thus, W = Ro(H/L)V, where H/L is the aspect ratio ≈ 7 km/1000 km = 0.007. This agrees with our earlier observation that the synoptic-scale vertical velocities are much less than the horizontal ones. Alternatively W/H = RoV/L. Since the terms are the same size, we keep them both. The dimensional QG continuity equation is almost the same as original pressure-coordinate version.

$$\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}+\frac{\partial \omega}{\partial p}=0$$

But all of the divergence is now carried by the ageostrophic wind.

The smallness of QG ω also changes the geostrophic Lagrangian derivative:

$$\frac{D_g}{Dt}=\frac{V}{L}\left\{\frac{\partial}{\partial t}+u_q\frac{\partial}{\partial x}+v_q\frac{\partial}{\partial y}\right\}+Ro\frac{V}{L}\left\{\omega\frac{\partial}{\partial p}\right\}\simeq\frac{V}{L}\left\{\frac{\partial}{\partial t}+u_q\frac{\partial}{\partial x}+v_q\frac{\partial}{\partial y}\right\}$$
$$=\frac{\partial}{\partial t}+u_q\frac{\partial}{\partial x}+v_q\frac{\partial}{\partial y}\equiv\frac{D_q}{Dt}$$

Here, we have redefined the QG Lagrangian derivative (q instead of g subscript) to neglect vertical advection. This new form also changes the QG momentum equation to:

$$\frac{D_q\mathbf{v}_q}{Dt}+f_0\mathbf{k}\times\mathbf{v}_a+\beta y\mathbf{k}\times\mathbf{v}_q=\frac{\partial \mathbf{v}_q}{\partial t}+u_q\frac{\partial \mathbf{v}_q}{\partial x}+v_q\frac{\partial \mathbf{v}_q}{\partial y}+f_0\mathbf{k}\times\mathbf{v}_a+\beta y\mathbf{k}\times\mathbf{v}_q=0$$

Finally, we address the Thermodynamic Energy Equation again, writing the temperature as $T_{total}=T_0(p)+T'(x,y,p,t)=T_0\{T_0+\mathrm{Ro}T'\}$. I.e, $\partial T_0/\partial p \gg \partial T/\partial p$, so that we can write:

$$\frac{\partial T}{\partial t}+u_q\frac{\partial T}{\partial x}+v_q\frac{\partial T}{\partial y}+\frac{T_0}{\theta_0}\frac{\partial \theta_0}{\partial p}\omega=\frac{J}{c_p}$$

Here we cannot neglect terms in ω because in a stable atmosphere vertical derivatives of temperature are much greater than horizontal ones.

Remember the hydrostatic equation, $\partial\phi/\partial p=-\alpha=-RT/p$. If we multiply the above equation by R/p, remembering that R is a constant and p is the vertical coordinate so that it can pass through all of the partial derivatives except ∂/∂p.

$$\frac{R}{p}\left(\frac{\partial T}{\partial t}+u_q\frac{\partial T}{\partial x}+v_q\frac{\partial T}{\partial y}\right)+\frac{R}{p}\frac{T_0}{\theta_0}\frac{\partial\theta_0}{\partial p}\omega=\frac{R}{p}\frac{J}{c_p}$$

$$\frac{\partial}{\partial t}\left(\frac{RT}{p}\right)+u_q\frac{\partial}{\partial x}\left(\frac{RT}{p}\right)+v_q\frac{\partial}{\partial y}\left(\frac{RT}{p}\right)+\frac{R}{p}\frac{T_0}{\theta_0}\frac{\partial\theta_0}{\partial p}\omega=\frac{R}{p}\frac{J}{c_p}$$

$$\frac{\partial}{\partial t}\left(-\frac{\partial\phi}{\partial p}\right)+u_q\frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial p}\right)+v_q\frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial p}\right)+\frac{R}{p}\frac{T_0}{\theta_0}\frac{\partial\theta_0}{\partial p}\omega=\frac{R}{p}\frac{J}{c_p}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial p}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial p}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial p}\right)+\sigma\omega=-\kappa\frac{J}{p}$$

Where $\kappa=R/c_p$ and

$$\sigma=-\frac{R}{p}\frac{T_0}{\theta_0}\frac{\partial\theta_0}{\partial p}=-\frac{\alpha_0}{\theta_0}\frac{\partial\theta_0}{\partial p}$$

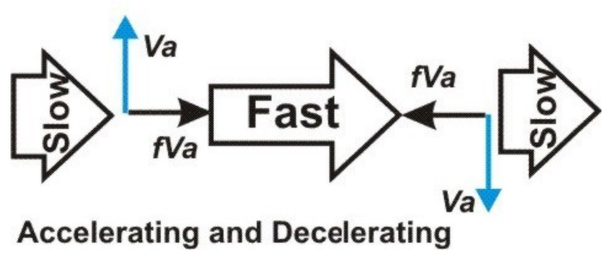
Because of the explicit negative sign in the definition of σ above and potential temperature's decrease downward (i.e. toward higher pressure), $\sigma>0$. Thus, upward motion (ω < 0, toward lower pressure) implies that –σω is positive, such that geopotential increases more slowly with decreasing pressure. Heating, positive *J*, will cause negative $D(\partial\phi/\partial p)/Dt$ in which geopotenrial becomes larger more quickly with decreasing pressure over time.

Some Applications

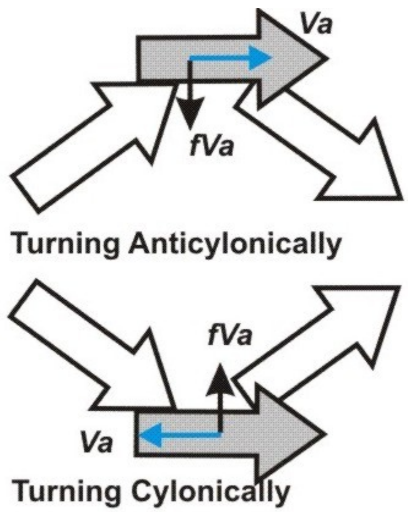
For now, let's neglect the β term in the QG momentum equation:

$$\frac{D_q\mathbf{v}_q}{Dt}+f_0\mathbf{k}\times\mathbf{v}_a+\beta y\mathbf{k}\times\mathbf{v}_q\approx\frac{D_q\mathbf{v}_q}{Dt}+f_0\mathbf{k}\times\mathbf{v}_a=0$$

With this approximation, acceleration of the geostrophic wind (for example in response to a tighter pressure gradient downstream) requires an ageostrophic wind that blows perpendicular to the geostrophic wind toward the left of the acceleration. It is reasonable to say that the wind will adjust to the pressure gradient and that the ageostrophic wind is the key to the adjustment process. Still, the <u>cause</u> of the changes in the geostrophic wind lie in its tendency to follow the pressure gradient and the agesotrophic wind is a result of the adjustment process.



When the straight-line wind is accelerating, the ageoptrophic wind blows from right to left (in the N Hemisphere) across the geostrophic wind such that the resulting Coriolis force is parallel with the geostrophic wind. When the straight-line wind is decelerating, the ageostrophic wind blows from left to produce a Coriolis force antiparallel with the geostrophic wind.



When the flow is anticyclonically curved, the ageostophic wind is parallel with the geostrophic wind such that the Coriolis force bends the wind to the right downstream. In this situation the total wind is stronger than the geostophic wind.

When the flow is cyclonically curved, the ageostophic wind is antiparallel with the geostrophic wind such that the Coriolis force bends the wind to the left downstream. In this situation the total wind is weaker than the geostophic wind.

These ageostrophic flows are very useful to understand the distributions of upper convergence and divergence around streaks of stronger wind in the upper tropospheric jet stream and around troughs and ridges on the jet.

The **Isallobaric Wind** is another example of ageostrophic wind, this time found more commonly near the surface. Let us suppose that all of the change of the quasigeostrophic wind is local (i.e. no advection) so that the QJ momentum equation on an *f*-plane ($f = f_0$) becomes:

$$\frac{\partial \mathbf{v}_q}{\partial t} = -f_0 \mathbf{k} \times \mathbf{v}_a$$

But from the definition of the geostrophic wind, $\mathbf{v}_g = f_0^{-1} \mathbf{k} \times \nabla \phi$. Since the Coriolis parameter and the vertical unit vector are constant, the Eulerian time derivative is:

$$\frac{\partial \mathbf{v}_q}{\partial t} = f_0^{-1} \mathbf{k} \times \nabla \frac{\partial \phi}{\partial t} = -f_0 \mathbf{k} \times \mathbf{v}_a$$

Which rearranges to:

$$\mathbf{k} \times \left(\nabla \frac{\partial \phi}{\partial t} + f_0^2 \mathbf{v}_a \right) = 0$$

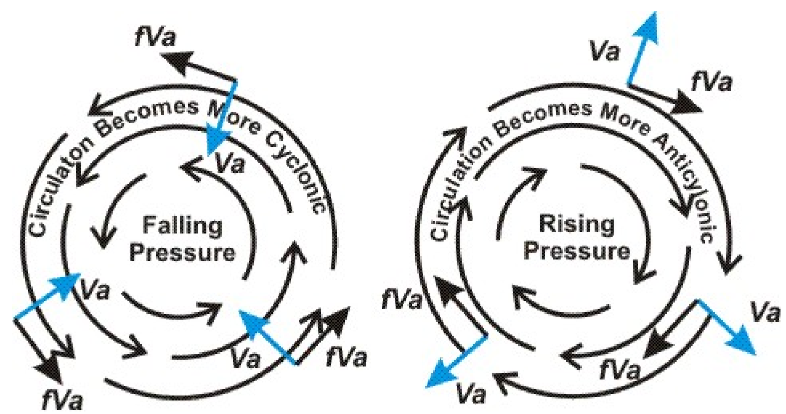
Since the wind and pressure gradient are horizontal (on a pressure surface), the cross product simply rotates the vector in parentheses by 90°. Thus, the only way to satisfy the equation is for the vector to be zero:

$$\nabla \frac{\partial \phi}{\partial t} = -f_0^2 \mathbf{v}_a$$

If we take $\nabla \cdot$ of the above,

$$\nabla \cdot \nabla \frac{\partial \phi}{\partial t} = \nabla^2 \frac{\partial \phi}{\partial t} = -f_0^2 \nabla \cdot \mathbf{v}_a$$

Thus, in a cell of pressure falls ($\nabla^2(\partial\phi/\partial t) > 0$) the ageostrophic wind will be convergent, i.e. negative divergence, and in a cell of pressure rises ($\nabla^2(\partial\phi/\partial t) < 0$) the ageostrophic wind will be divergent.



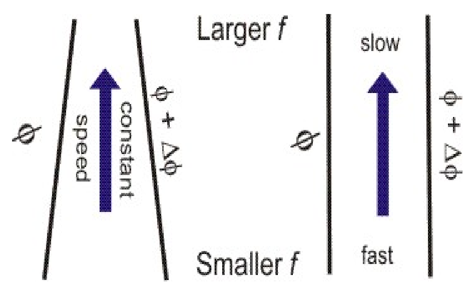
Finally let's look at the **Divergence of the <u>Geostrophic</u> Wind** (i.e. not the QG wind) to get some insight into the divergence of the QG wind.

$$\mathbf{v}_g = u_g\mathbf{i} + v_g\mathbf{j} = -\frac{1}{f}\frac{\partial\phi}{\partial y}\mathbf{i} + \frac{1}{f}\frac{\partial\phi}{\partial x}\mathbf{j}$$

The divergence is:

$$\nabla\bullet\mathbf{v}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{1}{f}\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial\phi}{\partial x}\right) = -\frac{1}{f}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{f}\frac{\partial^2\phi}{\partial x^2} - \frac{1}{f^2}\frac{\partial f}{\partial y}\frac{\partial\phi}{\partial x} = -\frac{\beta}{f^2}\frac{\partial\phi}{\partial x} = -\frac{\beta v_g}{f}$$

Thus, even a steady, straight-line geostrophic flow has a small divergence arising from <u>meridional</u> flow toward or away from higher or lower values of the Coriolis parameter. If v_g is constant poleward, the ϕ contours ---which are also streamlines---must become closer together poleward in order to keep the same speed as *f* increases; or if the ϕ gradient is constant v_g must slow poleward as *f* increases, and conversely for equatorward flow. This is the divergence of the ageostrophic wind due to β.



The QG Vorticity Equation:

Recall the definitions of the quasigeostrophic wind and relative vorticity:

$$u_q = -\frac{1}{f_0}\frac{\partial \phi}{\partial y}, \quad v_q = \frac{1}{f_0}\frac{\partial \phi}{\partial x}, \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Substituting the quasigeostrophic wind into the vorticity,

$$\zeta_q = \frac{\partial}{\partial x}\left(\frac{1}{f_0}\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{f_0}\frac{\partial \phi}{\partial y}\right) = \frac{1}{f_0}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2 \phi$$

Starting with the QG Momentum Equation in component form, make a vorticity equation:

$$-\frac{\partial}{\partial y}\bigg| \quad \frac{\partial u_q}{\partial t} + u_q \frac{\partial u_q}{\partial x} + v_q \frac{\partial u_q}{\partial y} - f_0 v_a - \beta y v_q = 0$$
$$\frac{\partial}{\partial x}\bigg| \quad \frac{\partial v_q}{\partial t} + u_q \frac{\partial v_q}{\partial x} + v_q \frac{\partial v_q}{\partial y} + f_0 u_a + \beta y u_q = 0$$

Carrying out the cross differentiation

$$\left(\frac{\partial}{\partial t} + u_q \frac{\partial}{\partial x} + v_q \frac{\partial}{\partial y}\right)\left(-\frac{\partial u_q}{\partial y}\right) - \frac{\partial u_q}{\partial y}\frac{\partial u_q}{\partial x} - \frac{\partial v_q}{\partial y}\frac{\partial u_q}{\partial y} + f_0 \frac{\partial v_a}{\partial y} + \beta v_q + \beta y \frac{\partial v_q}{\partial y} = 0$$
$$\left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}\right)\left(\frac{\partial v_g}{\partial x}\right) + \frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial v_g}{\partial y} + f_0 \frac{\partial u_a}{\partial x} + \quad \beta y \frac{\partial u_q}{\partial x} = 0$$

Adding:

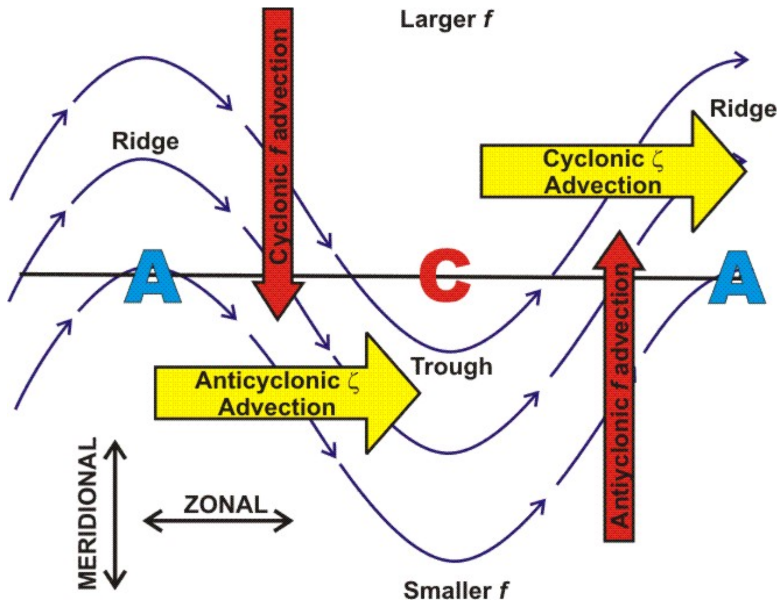
$$\left(\frac{\partial}{\partial t} + u_q \frac{\partial}{\partial x} + v_q \frac{\partial}{\partial y}\right)\left(\frac{\partial v_q}{\partial x} - \frac{\partial u_q}{\partial y}\right) + \left(\frac{\partial v_q}{\partial y} - \frac{\partial u_q}{\partial y}\right)\left(\frac{\partial u_q}{\partial x} + \frac{\partial v_q}{\partial y}\right) + f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right)$$
$$+\beta v_g + \beta y\left(\frac{\partial u_q}{\partial x} + \frac{\partial v_q}{\partial y}\right) = 0$$

But recall that the QG wind is nondivergent and all of the divergence is in the ageostrophic wind, so that the above simplifies to:

$$\left(\frac{\partial}{\partial t} + u_q \frac{\partial}{\partial x} + v_q \frac{\partial}{\partial y}\right)\left(\frac{\partial v_q}{\partial x} - \frac{\partial u_q}{\partial y}\right) + f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) + \beta v_q = 0$$

Recall from the QG continuity equation, $\partial u_a / \partial x + \partial v_a / \partial y = -\partial \omega / \partial p$ and that $D_q f / Dt = D_q (f_0 + \beta y) / Dt = \partial f / \partial t + u_q \partial f / \partial x + v_q \partial f / \partial y = 0 + 0 + v_q \partial f / \partial y$, so that the QG vorticity equation may be written:

$$\frac{D_q \zeta_q}{Dt} - f_0\left(\frac{\partial \omega}{\partial p}\right) + \beta v_q = \frac{D_q(\zeta_q + f)}{Dt} - f_0\left(\frac{\partial \omega}{\partial p}\right) = 0$$



Or,

$$\frac{\partial \zeta_q}{\partial t} = -v_q \cdot \nabla(\zeta_q + f) + f_0\left(\frac{\partial \omega}{\partial p}\right)$$
$$= -v_q \cdot \nabla \zeta_q - v_q \beta + f_0\left(\frac{\partial \omega}{\partial p}\right)$$

Wave-like perturbation on the zonal wind at 500 mb, with ∂ω/∂p small:

- ➔ Between a ridge and a downstream trough:
  - o Meridional advection of planetary vorticity by the north-to-south *v* component is positive, tending to move the trough upstream (toward the W).
  - o Zonal advection of relative vorticity by the west to east *u* component is negative, tending to move the tough downstream (toward the E)
- ➔ Between a trough and a downstream ridge:
  - o Meridional advection of planetary vorticity by the north-to-south *v* component is positive, tending to move the trough upstream (toward the W).
  - o Zonal advection of relative vorticity by the west to east *u* component is negative, tending to move the tough downstream (toward the E)

Imagine a situation where the geopotential is given by $\phi = -f_0 U_0 y + \phi_0 \cos(2\pi x / L_x)$. The meridional planetary vorticity advection that moves the pattern upstream is, $\beta v_g = \beta(2\pi\phi_0 / f_0 L_x)\sin(2\pi x / L_x)$, The

QG relative vorticity is $\zeta_g = -(4\pi^2\phi_0 / f_0 L_x^2)\cos(2\pi x / L_x)$; and the zonal relative advection that tends to move the pattern downstream is: $u_q \partial\zeta_q / \partial x \simeq U_0(-8\pi^3\phi_0 / f_0 L_x^3)\sin(2\pi x / L_x)$. For long (i.e. L_x large) waves, the planetary vorticity advection will generally be larger than the relative vorticity advection, and for short waves the relative vorticity advection will be larger.

Will the pattern be stationary when $\beta = 4\pi^2 U_0 / L_x^2$?

The **Geopotential Tendency Equation**: Start with the QG vorticity and thermodynamic energy equations:

$$\frac{1}{f_0}\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right) = -\vec{v}_q\cdot\nabla\left(\frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+f_0+\beta y\right)+f_0\left(\frac{\partial\omega}{\partial p}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial p}\right) = -\vec{v}_q\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right)-\sigma\omega-\frac{\kappa J}{p}$$

Switch the order of differentiation on the left side of both the vorticity and thermodynamic energy equations and multiply the latter by f_0/σ.

$$\frac{1}{f_0}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\phi}{\partial t} = -\vec{v}_q\cdot\nabla\left(\frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+f_0+\beta y\right)+f_0\left(\frac{\partial\omega}{\partial p}\right)$$

$$\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\frac{\partial\phi}{\partial t}\right) = -\frac{f_0}{\sigma}\vec{v}_q\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right)-f_0\omega-\frac{\kappa f_0 J}{\sigma p}$$

Differentiate the thermodynamic energy equation with respect to p:

$$\frac{f_0}{\sigma}\frac{\partial^2}{\partial p^2}\left(\frac{\partial\phi}{\partial t}\right) = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{v}_g\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right)\right]-f_0\frac{\partial\omega}{\partial p}-\frac{\kappa f_0}{\sigma}\frac{\partial}{\partial p}\left(\frac{J}{p}\right)$$

Add the modified thermodynamic equation to the vorticity equation canceling the $\partial\omega/\partial p$ term.

$$\frac{1}{f_0}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\frac{\partial\phi}{\partial t} = -\vec{v}_g\cdot\nabla\left(\frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+f_0+\beta y\right)$$
$$-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{v}_g\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right)\right]-\frac{\kappa f_0}{\sigma}\frac{\partial}{\partial p}\left(\frac{J}{p}\right)$$

The first term on the right is the absolute vorticity advection; the second is the vertical derivative of the geopotential thickness advection, and the last is the heating. The above relation is an example of a Poisson equation. As we have seen, it is readily solved for sinusoidally varying solutions and forcing. For example let's write $\partial\phi/\partial t \equiv \chi = X\cos(2\pi x/L_x + 2\pi y/L_y + 2\pi p/L_p)$ and write the forcing as a similarly structured periodic function $F\cos(2\pi x/L_x + 2\pi y/L_y + 2\pi p/L_p)$. Omitting the cosine function, which is the same on both sides of the equation,

$$-\frac{1}{f_0}\left[\frac{4\pi}{L_x^2}+\frac{4\pi}{L_y^2}+\frac{f_0^2}{\sigma}\frac{4\pi}{L_p^2}\right]X=F$$

So that the solution is just a scaled version of the forcing on the right side. There are also reliable and efficient numerical schemes to solve Poisson equations. The above relation shows that the solution is proportional to minus the forcing so that, for example, flow from cyclonic to anticyclonic vorticity will make the first term on the right positive such that it forces pressure falls. We like Poisson equations. Written in terms of *X,* the geopotential height tendency the equation is:

$$\frac{1}{f_0}\left[\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}+\frac{f_0^2}{\sigma}\frac{\partial^2\chi}{\partial p^2}\right]=-\vec{v}_g\cdot\nabla\left[\frac{1}{f_o}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+f_0+\beta y\right]-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{v}_g\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right)\right]-\frac{\kappa f_0}{\sigma}\frac{\partial}{\partial p}\left(\frac{J}{p}\right)$$

The height tendency equation is also a nice starting point to derive the QG potential vorticity equation. We assume adiabatic motion (J = 0), move the time derivative outside the three-dimensional Lapalacian, and expand the second term on the right.

$$\frac{1}{f_0}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\frac{\partial\phi}{\partial t}=\frac{\partial}{\partial t}\left[\frac{1}{f_0}\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{f_0}{\sigma}\frac{\partial^2\phi}{\partial p^2}\right]=$$
$$-\vec{v}_g\cdot\nabla\left(\frac{1}{f_o}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+f_0+\beta y\right)-\frac{f_0}{\sigma}\frac{\partial\vec{v}_g}{\partial p}\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right)-\vec{v}_g\cdot\nabla\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\phi}{\partial p}\right)',$$

But $f_0\partial\vec{v}_q/\partial p=\hat{k}\times\nabla\partial\phi/\partial p$, which is perpendicular to $\nabla(\partial\phi/\partial p)$ so that the second term on the right is zero, and

$$\frac{\partial}{\partial t}\left[\frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{f_0^2}{\sigma}\frac{\partial^2\phi}{\partial p^2}\right)\right]=-\vec{v}_q\cdot\nabla\left[\frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\frac{f_0}{\sigma}\frac{\partial^2\phi}{\partial p^2}+f_0+\beta y\right],$$

or

$$\frac{D_q q}{Dt}=\frac{\partial}{\partial t}\left[\frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{f_0^2}{\sigma}\frac{\partial^2\phi}{\partial p^2}\right)+f_0+\beta y\right]+\vec{v}_q\cdot\nabla\left[\frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{f_0^2}{\sigma}\frac{\partial^2\phi}{\partial p^2}\right)+f_0+\beta y\right]=0$$

Where $q=f_0^{-1}[\nabla^2\phi+(f_0^2/\sigma)\partial^2\phi/\partial p^2]+f_0+\beta y$ is the Quasigeostropic Potential Vorticity (QGPV). Note that the QGPV is related to ϕ through another one of those "easily solved" Poisson equations.

Derivation of the **Ω Equation** also begins with the thermodynamic and vorticity equations:

$$\frac{\partial}{\partial p}\left(\frac{\partial\phi}{\partial t}\right)=-\vec{v}_q\cdot\nabla\left(\frac{\partial\phi}{\partial t}\right)-\sigma\omega-\frac{\kappa J}{p}$$
$$\nabla^2\left(\frac{\partial\phi}{\partial t}\right)=-f_0\left[\vec{v}_q\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)\right]+f_0^2\left(\frac{\partial\omega}{\partial p}\right)$$

Take ∇^2 of the top equation and $\partial/\partial p$ of the bottom equation:

$$\nabla^2\left(\frac{\partial\phi}{\partial t\partial p}\right)=-\nabla^2\left[\vec{v}_q\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right)\right]-\sigma\nabla^2\omega-\kappa\nabla^2\left(\frac{J}{p}\right)$$

$$\frac{\partial}{\partial p}\left[\nabla^2\left(\frac{\partial\phi}{\partial t}\right)\right]=-f_0\frac{\partial}{\partial p}\left[\vec{v}_q\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)\right]+f_0^2\left(\frac{\partial^2\omega}{\partial p^2}\right)$$

Flipping the order of differentiation on the right in the top equation and subtracting it from the bottom:

$$0=-f_0\frac{\partial}{\partial p}\left[\vec{v}_q\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)\right]+f_0^2\left(\frac{\partial^2\omega}{\partial p^2}\right)+\nabla^2\left[\vec{v}_q\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right)\right]+\sigma\nabla^2\omega+\kappa\nabla^2\left(\frac{J}{p}\right)$$

Which rearranges to:

$$\nabla^2\omega+\frac{f_0^2}{\sigma}\left(\frac{\partial^2\omega}{\partial p^2}\right)=\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{v}_q\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)\right]+\frac{1}{\sigma}\nabla^2\left[\vec{v}_q\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]+\frac{\kappa}{\sigma}\nabla^2\left(\frac{J}{p}\right)$$

The expression on the left is the Laplacian of ω. The first term on the left is the vertical derivative of the absolute vorticity advection; the second term is the Laplacian of the temperature advection; and the last term on the left is the Laplacian of the heating. In practice, the first and second terms tend to cancel. Applying the product rule for differentiation to the first term on the right yields:

$$\frac{f_0}{\sigma}\left[\frac{\partial\vec{v}_q}{\partial p}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)+\vec{v}_q\cdot\nabla\left(\frac{1}{f_0}\nabla^2\frac{\partial\phi}{\partial p}\right)\right]$$

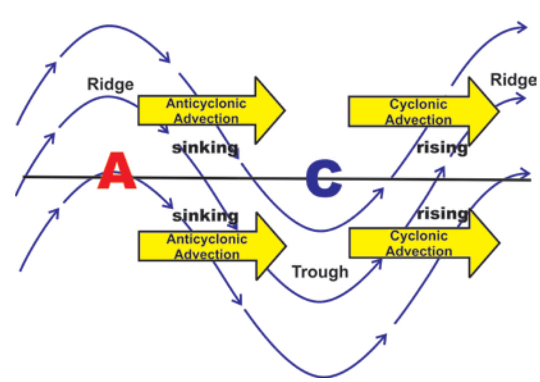
Applying the product rule to the second term yields:

$$-\frac{1}{\sigma}\left[\nabla^2u_q\frac{\partial}{\partial x}\left(\frac{\partial\nabla^2\phi}{\partial p}\right)+\nabla^2v_q\frac{\partial}{\partial y}\left(\frac{\partial\nabla^2\phi}{\partial p}\right)\right]-\frac{\vec{v}_q}{\sigma}\cdot\nabla\left(\frac{\partial\nabla^2\phi}{\partial p}\right)$$

Thus the last terms in these two expressions cancel, and (in turns out that) the first two terms in the second expression are small. The final (somewhat approximate) version of the Ω Equation is:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{f_0^2}{\sigma}\frac{\partial^2\omega}{\partial p^2}\right)\omega\simeq\frac{f_0}{\sigma}\frac{\partial\vec{v}_q}{\partial p}\bullet\nabla\left[\frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+f_0+\beta y\right]$$

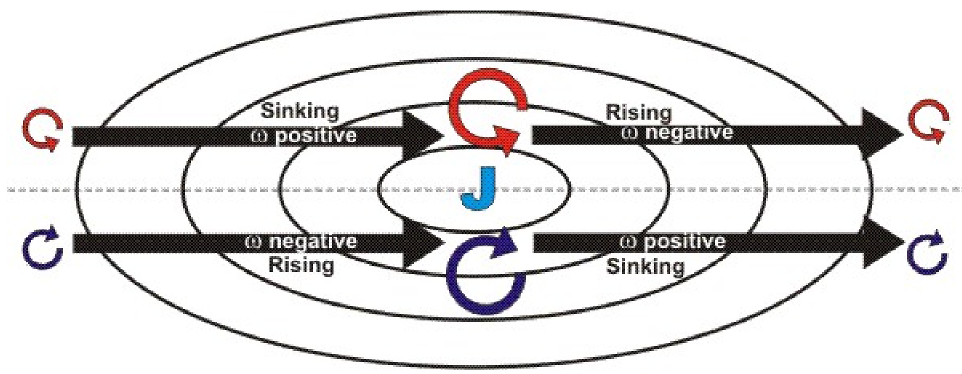
The forcing here is the advection of the absolute vorticity by the thermal wind. When a westerly geostrophic wind increases upward, $\partial\vec{v}_q/\partial p<0$, and blows from anticyclonic toward cyclonic vorticity so that the vorticity increases downstream, the forcing is negative, implying positive ω (sinking motion) between the upper ridge and upper trough. Conversely, when the same thermal wind blows from cyclonic toward anticyclonic upper vorticity, the forcing is positive, implying negative ω (rising motion)



between the trough and the ridge. Since the surface low is generally between the upper trough and upper ridge and surface high is between the upper ridge and upper trough, rising motion overlies the surface low and sinking motion overlies the surface high.

More concisely, rising motion ($\omega < 0$) arises when thermal wind that increases upward advects cyclonic vorticity to produce positive forcing on the right side of the ω equation; sinking motion ($\omega > 0$) arises when thermal wind that increases upward advects anticyclonic vorticity to produce negative forcing on the right side of the ω equation.

The classic 4-quadrant model of a straight Jet Streak is another illustration of vertical motion as deduced from the ω equation. Remember that the Jet Streak moves more slowly than the wind at the level of the Jet. As air enters the streak from the upstream (west) end and exits from the downstream (east) end it first accelerates and then decorates to remain in geostrophic balance with the tightening and then relaxing pressure gradiant.



- In the **right entrance** region, the jet aloft advects relatively anticyclonic vorticity toward the maximum causing positive forcing and **rising** motion ($\omega < 0$).
- In the **left entrance** region, the jet aloft advects relatively cyclonic vorticity toward the maximum causing positive forcing and **sinking** motion ($\omega > 0$).
- In the **right exit** region, the jet aloft advects relatively cyclonic vorticity away from the maximum causing negative forcing and **sinking** motion ($\omega > 0$).
- In the **left exit** region, the jet aloft advects relatively cyclonic vorticity away from the maximum causing negative forcing and **rising** motion ($\omega < 0$).