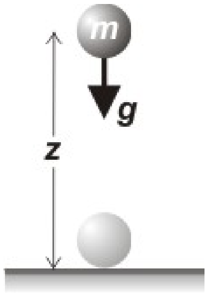
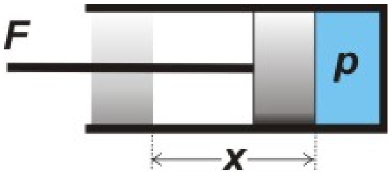
Objective: To introduce energy equations in the geostrophic and quasigeostrophic context

Reading: CH 2: 46-49

Problem: At the end of these notes



Mechanical Energy is produced by a force acting through a distance. For example, raising an object of m through a distance z, against the gravitational acceleration imparts ***Potential Energy*** = *mgz* to the object. While the object is being raised with vertical velocity w, the applied force is doing work at a rate *mgw*. Another example of mechanical work is a (variable) force *F* compressing a gas with pressure *p* by moving a piston with surface area *A* a distance *x* into a cylinder. Here, *F* = *pA* and the work done is ∫*F*d*x* = ∫*Ap*(*x*)d*x*. ***Kinetic Energy*** = mv^2/2 represents the energy of motion when an object of mass m moves with velocity v. Mechanical energy is measured in Joules (kg m^2 s^{-2}), so that specific energy (energy per unit mass) has units of velocity squared [(m s



Another form of energy is thermodynamic energy, of which the most commonly encounterd form is Enthalpy = $E = c_pT$. Changes in enthalpy represent changes in temperature, *T*, as a result of adding or removing heat at constant pressure, allowing the volume to change. We have already dealt with the thermodynamic energy equation. The meteorological units for c_p, the specific heat at constant pressure are J kg^{-1}K^{-1}. Thus Enthalpy per unit volume also has units of velocity squared.

In adiabatic vertical motion Enthalpy is converted into gravitational potential energy $c_pm\Delta T = mg\Delta z$, or $c_p\Delta T/\Delta z = g/c_p$ to produce the dry adiabatic lapse rate.

Here we are interested in computing the rate of change of the wind's kinetic energy as a result of the work done by the forces on the right-hand side of the momentum equations. The basis of the analysis is taking the vector inner product of the velocity components with the momentum equations.

We start with quasigeostrophic system with the geopotential expressed explicitly:

$$\frac{\partial u_q}{\partial t} + u_q\frac{\partial u_q}{\partial x} + v_q\frac{\partial u_q}{\partial y} - f_0 v_a - \beta y v_q = 0$$

$$\frac{\partial v_q}{\partial t} + u_q\frac{\partial v_q}{\partial x} + v_q\frac{\partial v_q}{\partial y} + f_0 u_a + \beta y u_q = 0$$

Multiplying u_q times the *x* momentum equation and v_q times the *y* momentum equation,

$$\frac{\partial}{\partial t}\frac{u_q^2}{2}+u_q\frac{\partial}{\partial x}\frac{u_q^2}{2}+v_q\frac{\partial}{\partial y}\frac{u_q^2}{2}-f_0u_qv_a-\beta yu_qv_q=0$$
$$\frac{\partial}{\partial t}\frac{v_q^2}{2}+u_q\frac{\partial}{\partial x}\frac{v_q^2}{2}+v_q\frac{\partial}{\partial y}\frac{v_q^2}{2}+f_0v_qu_a+\beta yv_qu_q=0$$

Adding,

$$\frac{\partial}{\partial t}\left(\frac{u_q^2}{2}+\frac{v_q^2}{2}\right)+u_q\frac{\partial}{\partial x}\left(\frac{u_q^2}{2}+\frac{v_q^2}{2}\right)+v_q\frac{\partial}{\partial y}\left(\frac{u_q^2}{2}+\frac{v_q^2}{2}\right)=\frac{D}{Dt}\left(\frac{u_q^2}{2}+\frac{v_q^2}{2}\right)=$$
$$f_0u_qv_a-f_0v_qu_a+\beta y(u_qv_q-v_qu_q)=-v_a\frac{\partial\phi}{\partial y}-u_a\frac{\partial\phi}{\partial x}=-u_a\frac{\partial\phi}{\partial x}-v_a\frac{\partial\phi}{\partial y}.$$

Thus, in the QG formalism, the Lagrangian change of the wind's kinetic energy [$\frac{1}{2}(u_q^2+v_q^2)$] is proportional to generation by the ageostrophic wind blowing down the geopotential (ϕ) gradient. To answer this question we consider that the QG wind is in strict geostrophic balance, $f_0\mathbf{k}\times\mathbf{v}_q=-\nabla\phi$. Taking the outer product of **k** with this expression yields $\mathbf{v}_q = f_0^{-1}\mathbf{k}\times\nabla\phi$, which vector is perpendicular to $\nabla\phi$ so that the advection of the geopotential by the QG wind is in fact zero and the QG energy equation simply generation of kinetic energy by ageostrophic cross contour flow.

To get a more realistic picture of mechanical energy in the atmosphere, we consider the primitive equations in pressure coordinates without making the QG approximations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial p}-fv=-\frac{\partial\phi}{\partial x}$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial v}{\partial p}+fu=-\frac{\partial\phi}{\partial y}$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial\omega}{\partial p}=0$$

As before, we multiply the *x* momentum equation by *u* and the *y* momentum equation by *v*.

$$\frac{\partial}{\partial t}\frac{u^2}{2}+u\frac{\partial}{\partial x}\frac{u^2}{2}+v\frac{\partial}{\partial y}\frac{u^2}{2}+\omega\frac{\partial}{\partial p}\frac{u^2}{2}-fuv=-u\frac{\partial\phi}{\partial x}$$
$$\frac{\partial}{\partial t}\frac{v^2}{2}+u\frac{\partial}{\partial x}\frac{v^2}{2}+v\frac{\partial}{\partial y}\frac{v^2}{2}+\omega\frac{\partial}{\partial p}\frac{v^2}{2}+fuv=-v\frac{\partial\phi}{\partial y}$$

And add

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}+\frac{v^2}{2}\right)+u\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}\right)+v\frac{\partial}{\partial y}\left(\frac{u^2}{2}+\frac{v^2}{2}\right)+\omega\frac{\partial}{\partial p}\left(\frac{u^2}{2}+\frac{v^2}{2}\right)=-u\frac{\partial\phi}{\partial x}-v\frac{\partial\phi}{\partial y}$$

We write $K=\frac{1}{2}(u_q^2+v_q^2)$ so that

$$\frac{\partial K}{\partial t}+u\frac{\partial K}{\partial x}+v\frac{\partial K}{\partial y}+\omega\frac{\partial K}{\partial p}=-u\frac{\partial\phi}{\partial x}-v\frac{\partial\phi}{\partial y}.$$

Taking *K* times mass continuity and recognizing that $-u\partial\phi/\partial x = -\partial u\phi/\partial x + \phi\partial u/\partial x$ and $-v\partial\phi/\partial y = -\partial v\phi/\partial y + \phi\partial v/\partial y$ produces:

$$\frac{\partial K}{\partial t}+\frac{\partial(uK)}{\partial x}+\frac{\partial(vK)}{\partial y}+\frac{\partial(\omega K)}{\partial p}=-\frac{\partial u\phi}{\partial x}+\phi\frac{\partial u}{\partial x}-\frac{\partial v\phi}{\partial y}+\phi\frac{\partial v}{\partial y}$$

Which simplifies to:

$$\frac{\partial K}{\partial t}+\frac{\partial[u(K+\phi)]}{\partial x}+\frac{\partial[v(K+\phi)]}{\partial y}+\frac{\partial(\omega K)}{\partial p}=\phi\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$
$$=-\phi\frac{\partial\omega}{\partial p}=-\frac{\partial\phi\omega}{\partial p}+\omega\frac{\partial\phi}{\partial p}$$

Rearranging to get a kinetic energy equation

$$\frac{\partial K}{\partial t}+\frac{\partial[u(K+\phi)]}{\partial x}+\frac{\partial[v(K+\phi)]}{\partial y}+\frac{\partial[\omega(K+\phi)]}{\partial p}=-\frac{RT}{p}\omega$$

Remember the thermodynamic energy equation:

$$c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+\omega\frac{\partial T}{\partial p}\right)-\alpha\omega=J$$

Since $c_pT = E$ = enthalpy and $\alpha = RT/p$, we can add enthalpy times the mass continuity equation and rearrange:

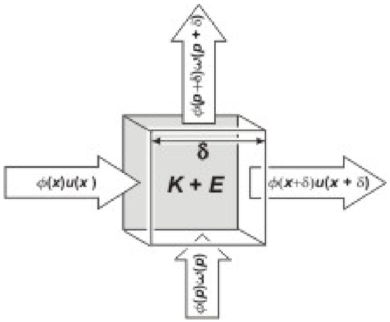
$$\frac{\partial E}{\partial t}+\frac{\partial(uE)}{\partial x}+\frac{\partial(vE)}{\partial y}+\frac{\partial(\omega E)}{\partial p}=\frac{RT}{p}\omega+J$$

Adding the kinetic energy and thermodynamic energy equations yields,

$$\frac{\partial(K+E)}{\partial t}+\frac{\partial[u(K+E+\phi)]}{\partial x}+\frac{\partial[v(K+E+\phi)]}{\partial y}+\frac{\partial[\omega(K+E+\phi)]}{\partial p}=J$$

Another form of the same equation is:

$$\frac{\partial(K+E)}{\partial t}+\frac{\partial[u(K+E)]}{\partial x}+\frac{\partial[v(K+E)]}{\partial y}+\frac{\partial[\omega(K+E)]}{\partial p}=
-\frac{\partial(u\phi)}{\partial x}-\frac{\partial(v\phi)}{\partial y}-\frac{\partial(\omega\phi)}{\partial p}+J$$



This equation is interpreted as the (flux form) individual derivative of kinetic plus thermodynamic energy on the left set equal to the flux convergence geopotential work plus diabatic heating on the right.

A glance at the 500 mb chart shows that typical wind speed fluctuations are about 30 m s^{-1}, so that variations in K are about 450 m^2 s^{-2}. Corresponding changes of 500 mb height are ~500 m and of temperature are ~30^{o}C, so that the fluctuations of geopotential and enthalpy are 5 x 10^3 m^2 s^{-2} and 3 x 10^4 m^2 s^{-2} respectively. As we showed above the geostrophic flow tends to parallel the height contours so that conversion among these forms of energy is inefficient. This assertion is borne out by the ratios of kinetic energy to geopotential, one to ten, and that of kinetic energy to enthalpy, one to sixty.

Problem:

As discussed earlier, bulk aerodynamic friction is expressed as $\tau = -C_D\rho V^2$, where τ is the frictional stress per unit area (in Nt m^{-2}), C_D is a drag coefficient = 2x10^{-3} over open water, ρ = 1.1 kg m^{-3} is the air density, and V is the wind speed at the 10 m reference level. Derive and expression for the rate at which the wind does work against friction. If the 10 m wind is 50 m s^{-1}, what is the power dissipation? How does this value compare with 250 W m^{-2}, the globally averaged solar heating rate?