

**Objective:** To introduce the shallow-water model of geophysical fluid motions

**Topics:**

- Shallow-water governing equations
- Shallow-water gravity waves
- Shallow-water vorticity and potential vorticity equations
- Shallow-water energy equation

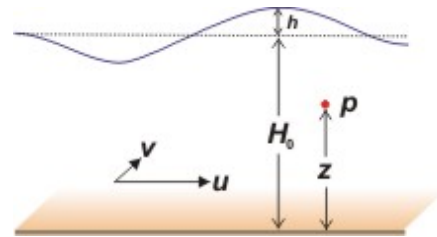
**Reading:** CH 7, pp 192-195

**Problems:** NONE

### Shallow-Water Equations

Consider a shallow, incompressible fluid with resting depth,  $H_0$ , and constant density,  $\rho$ . Its motions are governed by the momentum, hydrostatic, and mass continuity equations:

$$\begin{aligned}\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial x}, \\ \frac{Dv}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} &= -g\rho \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0\end{aligned}$$



If the fluid is barotropic and shallow enough,  $u$  and  $v$  will be constant functions of depth. If  $h$  is the displacement of the surface from its resting level, the vertical velocity at the surface is  $w(H_0 + h) = D(H_0 + h) / Dt \cong Dh / Dt$ . If the bottom is flat and impervious,  $w(0) = 0$ . Integrating the mass continuity equation from the bottom to the surface produces:

$$\begin{aligned}\int_0^{H_0+h} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz &= (H_0 + h) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + w(H_0 + h) - w(0) \\ &= (H_0 + h) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{D(H_0 + h)}{Dt} - 0 \approx H_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{Dh}{Dt} = 0\end{aligned}$$

From the hydrostatic law, the pressure at a sensor at height  $z$  above the bottom is,

$$p = g\rho(H_0 + h - z)$$

The partial derivatives of the pressure with respect to  $t$ ,  $x$ ,  $y$ , and its individual derivative are:

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{\partial}{\partial t} g\rho(H_0 + h - z) = g\rho \frac{\partial h}{\partial t}, \\ \frac{\partial p}{\partial x} &= \frac{\partial}{\partial x} g\rho(H_0 + h - z) = g\rho \frac{\partial h}{\partial x}, \\ \frac{\partial p}{\partial y} &= \frac{\partial}{\partial y} g\rho(H_0 + h - z) = g\rho \frac{\partial h}{\partial y}, \\ \frac{Dp}{Dt} &= g\rho \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right) = -g\rho H_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),\end{aligned}$$

since  $H_0$  and  $z$  are constant. The momentum equations then become:

$$\begin{aligned}\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} = -g \frac{\partial h}{\partial x} \\ \frac{Dv}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} = -g \frac{\partial h}{\partial y}\end{aligned}$$

### Shallow-Water Gravity Waves

For simplicity, assume that we can neglect the advective terms in the governing equations.

$$\begin{aligned}\frac{\partial u}{\partial t} &= -g \frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} &= -g \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} &= -H_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)\end{aligned}$$

Take the time derivative of the continuity equation,

$$\begin{aligned}\frac{\partial^2 h}{\partial t^2} &= -\frac{\partial}{\partial t} H_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -H_0 \left[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial t} \right) \right] \\ &= -H_0 \left[ \frac{\partial}{\partial x} \left( -g \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( -g \frac{\partial h}{\partial y} \right) \right] \\ &= gH_0 \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right)\end{aligned}$$

Assume sinusoidal (imaginary exponential) solutions for waves with period  $T$ , wavelength in the  $x$  and  $y$  directions  $L_x$  and  $L_y$ , of the form  $h = A \cos 2\pi(x/L_x + y/L_y - t/T)$  where  $A$  is the (constant) Amplitude.

Substituting this solution into the above equation produces:

$$-A \left( \frac{2\pi}{T} \right)^2 \cos 2\pi \left( \frac{x}{L_x} + \frac{y}{L_y} - \frac{t}{T} \right) = gH_0 A \left[ - \left( \frac{2\pi}{L_x} \right)^2 - \left( \frac{2\pi}{L_y} \right)^2 \right] \cos 2\pi \left( \frac{x}{L_x} + \frac{y}{L_y} - \frac{t}{T} \right),$$

$$c^2 = \frac{(1/T)^2}{(1/L_x)^2 + (1/L_y)^2} = \frac{T^{-2}}{L_x^{-2} + L_y^{-2}} = gH_0$$

Because A and the cosine function itself cancel out, leaving us with a **Dispersion Relation**, that we can solve for the waves' **Phase Velocity**. These are shallow-water gravity waves. For the approximations that we have made to be accurate, the total wavelength,  $2\pi/(L_x^{-2} + L_y^{-2})^{1/2}$ , needs to be much greater than the depth and the displacement of the surface must be much less than the resting depth. In that case, the phase speed,  $\pm(gH)^{1/2}$ , depends only upon the depth. Here are typical values:

$$\text{For 10 m depth, } (gH)^{1/2} = [(9.8 \text{ m s}^{-2}) * (10 \text{ m})]^{1/2} = 9.9 \text{ m s}^{-1}$$

$$\text{For 100 m depth, } [(9.8 \text{ m s}^{-2}) * (100 \text{ m})]^{1/2} = 31.3 \text{ m s}^{-1}$$

$$\text{For 1000 m depth, } [(9.8 \text{ m s}^{-2}) * (1000 \text{ m})]^{1/2} = 99 \text{ m s}^{-1}$$

$$\text{For 4000 m depth, } [(9.8 \text{ m s}^{-2}) * (4000 \text{ m})]^{1/2} = 198 \text{ m s}^{-1}$$

#### Shallow water vorticity and potential vorticity equations:

To look at shallow-water motions more closely, let's make a shallow-water vorticity equation from the momentum and continuity equations. Here, we write the total depth,  $H = H_0 + h$  and step back from approximating the depth as the resting depth and the derivatives of depth as derivatives of the surface's displacement,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv + g \frac{\partial H}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu + g \frac{\partial H}{\partial y} = 0 \quad ,$$

$$\frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

As before, we take  $-\partial/\partial y$  of the zonal momentum equation and  $\partial/\partial x$  of the meridional momentum equation and add them together:

$$\frac{\partial}{\partial t} \left( -\frac{\partial u}{\partial y} \right) + u \frac{\partial}{\partial x} \left( -\frac{\partial u}{\partial y} \right) - \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + v \frac{\partial}{\partial y} \left( -\frac{\partial u}{\partial y} \right) - \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + f \frac{\partial v}{\partial y} + \frac{\partial f}{\partial y} v - g \frac{\partial^2 H}{\partial x \partial y} = 0$$

$$\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} \right) + u \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} \right) + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + v \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + f \frac{\partial u}{\partial x} + g \frac{\partial^2 H}{\partial x \partial y} = 0$$

$$\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + u \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + v \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\partial f}{\partial y} v + 0 = 0$$

Substituting from mass continuity, and recognizing that  $fv = Df/Dt$ :

$$\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right) + u \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right) + v \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right) - \frac{1}{H} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right) \left( \frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} \right) = 0$$

If we write the relative vorticity,  $\zeta = \partial v/\partial x - \partial u/\partial y$ , and recognize the Lagrangian derivatives,

$$\frac{D(\zeta + f)}{Dt} - \frac{\zeta + f}{H} \frac{DH}{Dt} = 0$$

Dividing by  $H$  and using the quotient rule from calculus:

$$\frac{1}{H} \frac{D(\zeta + f)}{Dt} - \frac{\zeta + f}{H^2} \frac{DH}{Dt} = \frac{D}{Dt} \left( \frac{\zeta + f}{H} \right) = 0$$

Thus, the **Shallow-Water Potential Vorticity**,  $(\zeta + f)/H$ , is a conserved quantity, just as the QG potential vorticity is. For completeness, let's rewrite the vorticity equation one more time:

$$\frac{D\zeta}{Dt} - \frac{\zeta + f}{H} \frac{DH}{Dt} + \beta v = 0,$$

where  $\beta = \partial f/\partial y$ . If we make the Shallow-Water QG approximations, such that  $f \approx f_0 \gg \zeta$ ,  $\beta y \ll f_0$ , the wind and vorticity are geostrophic with constant  $f_0$  and  $H_0 \gg h$ ,

$$\frac{D}{Dt} \frac{g}{f_0} \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) - \frac{f_0}{H_0} \frac{Dh}{Dt} + \beta \frac{g}{f_0} \frac{\partial h}{\partial x} = 0$$

Which simplifies to,

$$\frac{D}{Dt} \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) - \frac{f_0^2}{gH_0} \frac{Dh}{Dt} + \beta \frac{\partial h}{\partial x} = 0$$

This is the **QG Shallow Water Vorticity Equation**.

### Shallow Water Energy Equation

Start with the shallow-water primitive equations:

$$\begin{aligned}\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv + g \frac{\partial H}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu + g \frac{\partial H}{\partial y} &= 0 \\ \frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0\end{aligned}$$

Multiplying the momentum equations by  $u$  and  $v$ , as before:

$$\begin{aligned}\frac{\partial}{\partial t} \frac{u^2}{2} + u \frac{\partial}{\partial x} \frac{u^2}{2} + v \frac{\partial}{\partial y} \frac{u^2}{2} - fuv &= -gu \frac{\partial H}{\partial x} \\ \frac{\partial}{\partial t} \frac{v^2}{2} + u \frac{\partial}{\partial x} \frac{v^2}{2} + v \frac{\partial}{\partial y} \frac{v^2}{2} + fuv &= -gv \frac{\partial H}{\partial y}\end{aligned}$$

Adding the momentum equations,

$$\frac{\partial}{\partial t} \left( \frac{u^2}{2} + \frac{v^2}{2} \right) + u \frac{\partial}{\partial x} \left( \frac{u^2}{2} + \frac{v^2}{2} \right) + v \frac{\partial}{\partial y} \left( \frac{u^2}{2} + \frac{v^2}{2} \right) - 0 = -g \left( u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} \right)$$

and then  $g$  times the mass continuity equation:

$$\begin{aligned}\frac{\partial}{\partial t} \left( \frac{u^2}{2} + \frac{v^2}{2} + gH \right) + u \frac{\partial}{\partial x} \left( \frac{u^2}{2} + \frac{v^2}{2} + gH \right) + v \frac{\partial}{\partial y} \left( \frac{u^2}{2} + \frac{v^2}{2} + gH \right) \\ = -g \left( u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} \right) - gH \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -g \left( \frac{\partial(uH)}{\partial x} + \frac{\partial(vH)}{\partial y} \right),\end{aligned}$$

which may be rewritten as the **Shallow-Water Energy Equation** in advective form:

$$\frac{\partial}{\partial t} (K + gH) + u \frac{\partial}{\partial x} (K + gH) + v \frac{\partial}{\partial y} (K + gH) = -g \left( \frac{\partial(uH)}{\partial x} + \frac{\partial(vH)}{\partial y} \right)$$

Take  $H$  times the above equation and  $(K + gH)$  times continuity:

$$\begin{aligned}H \frac{\partial}{\partial t} (K + gH) + uH \frac{\partial}{\partial x} (K + gH) + vH \frac{\partial}{\partial y} (K + gH) &= -gH \left( \frac{\partial(uH)}{\partial x} + \frac{\partial(vH)}{\partial y} \right) \\ (K + gH) \frac{\partial H}{\partial t} + (K + gH) \frac{\partial uH}{\partial x} + (K + gH) \frac{\partial vH}{\partial y} &= 0\end{aligned}$$

Add to get the energy equation in flux form:

$$\frac{\partial}{\partial t} H(K + gH) + \frac{\partial}{\partial x} uH(K + gH) + \frac{\partial}{\partial y} vH(K + gH) = -gH \left( \frac{\partial(uH)}{\partial x} + \frac{\partial(vH)}{\partial y} \right)$$

The left hand side is the flux-form individual derivative of the column integrated kinetic and potential energy, and the right-hand side is the column integrated pressure work. One could recognize that the right side is  $gH(\partial H/\partial t) = g (\frac{1}{2}\partial H^2/\partial t)$  so that,

$$\frac{\partial}{\partial t} \left( HK + \frac{gH^2}{2} \right) + \frac{\partial}{\partial x} uH(K + gH) + \frac{\partial}{\partial y} vH(K + gH) = 0$$