Objectives:

- To understand vertical motion in unstable baroclinic waves
- To understand the energetic of unstable baroclinic waves
- To understand the atmospheric energy (Lorenz) cycle

Reading: Holton, pp238-250

Problem: Continue from 3A

Vertical Motion in an Unstable Baroclinic Wave

Consider the omega equation that we derived previously:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \approx \frac{f_0}{\sigma}\frac{\partial \mathbf{v}_g}{\partial p}\bullet\nabla\left(\frac{1}{f_0}\nabla^2\phi+f\right)$$

We represent the second partial derivative with respect to pressure as,

$$\frac{\partial^2\omega}{\partial p^2}\approx\frac{(\partial\omega/\partial p)_3-(\partial\omega/\partial p)_1}{\delta p}=-\frac{2\omega_2}{(\delta p)^2}$$

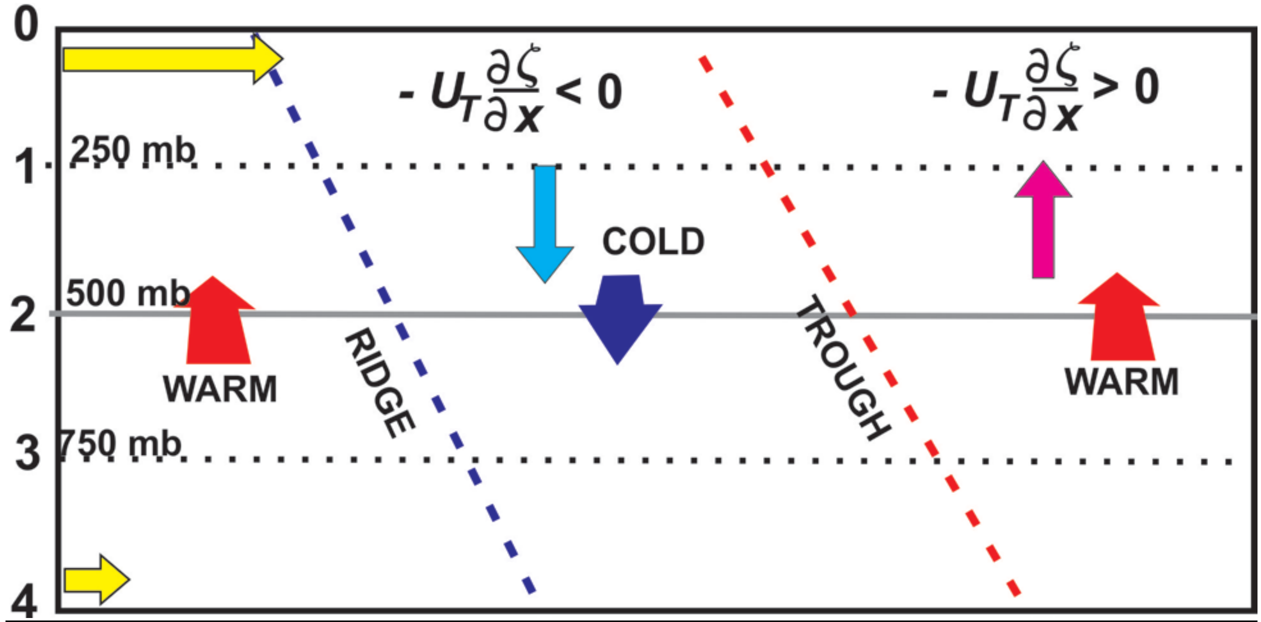
So that for one-dimensional disturbances with zonal wavenumber *k*,

$$-\left(k^2+\frac{2f_0^2}{\sigma(\delta p)^2}\right)\omega=\frac{f_0}{\sigma}\frac{U_3-U_1}{\delta p}\frac{\partial}{\partial x}\frac{\partial^2\psi_2}{\partial x^2}=-\frac{2f_0}{\sigma}\frac{U_1-U_3}{2}\frac{1}{\delta p}\frac{\partial}{\partial x}\frac{\partial^2\psi_2}{\partial x^2}=-\frac{2f_0}{\sigma}\frac{U_T}{\delta p}\frac{\partial\zeta_T}{\partial x}$$

Or,

$$\omega=\left(k^2+\frac{2f_0^2}{\sigma(\delta p)^2}\right)^{-1}\frac{2f_0}{\sigma}\frac{U_T}{\delta p}\frac{\partial\zeta_T}{\partial x}$$

Thus, when the thermal wind blows for cyclonic perturbation vorticity to anticyclonic perturbation vorticity $\partial\zeta_T/\partial x<0$ so that ω < 0, implying rising motion. Conversely, when the thermal wind blows from anticyclonic to cyclonic vorticity $\partial\zeta_T/\partial x>0$ so that ω > 0, implying rising motion. Therefore, sinking motion fills the volume between the ridge and the downstream trough, and rising motion fills the volume between the trough and the downstream ridge.



Energy transforms in the two-layer model:

Consider the two-layer governing equations from before:

$$\left(\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_1}{\partial x^2}+\beta\frac{\partial\psi_1}{\partial x}=\frac{f_0}{\delta p}\omega_2,$$
$$\left(\frac{\partial}{\partial t}+U_3\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_3}{\partial x^2}+\beta\frac{\partial\psi_3}{\partial x}=-\frac{f_0}{\delta p}\omega_2,$$
$$\left(\frac{\partial}{\partial t}+U_M\frac{\partial}{\partial x}\right)(\psi_1-\psi_3)-U_T\frac{\partial}{\partial x}(\psi_1+\psi_3)=\frac{\sigma\delta p}{f_0}\omega_2$$

These are the vorticity equations for the upper and lower layer, coupled by vorticity stretching due to the pressure coordinate vertical velocity in the middle layer, and the thermodynamic energy equation applied to the middle layer.

Recall that $\psi_1 - \psi_3$ is proportional to the temperature in the middle layer and that these are perturbation streamfunction. The total QG streamfunction is $\psi = \phi/f_0$, so that the perturbation streamfunctions used here are $\psi_1 = f_0^{-1}(\phi_1-\overline{\phi}_1)$ and $\psi_2 = f_0^{-1}(\phi_2-\overline{\phi}_2)$. From the Hypsometeric equation $\phi_1-\phi_3 = RT_2\ln(p_3/p_1)$ and $\overline{\phi}_1-\overline{\phi}_3 = R\overline{T}_2\ln(p_3/p_1)$. Combining these relations,

$$\psi_1-\psi_3=\frac{1}{f_0}[\phi_1-\phi_3-(\overline{\phi}_1-\overline{\phi}_3)]=\frac{R(T_2-\overline{T}_2)}{f_0}\ln\frac{p_3}{p_1}$$

Similarly, $U_T = (R / f_0)(\partial \overline{T}_2 / \partial y)\ln(p_3 / p_2)$ from the Thermal Wind Equation. Recall from the derivation of the thermodynamic energy equation that $\sigma = (R\overline{T} / p_2)\overline{\theta}^{-1}\partial\overline{\theta} / \partial p = (gH / p_2)\overline{\theta}^{-1}\partial\overline{\theta} / \partial p$ $= (\partial\overline{\phi} / \partial p)\overline{\theta}^{-1}(\partial\overline{\theta} / \partial p)$.

Here we will focus on computing the integrated energy transformations over a distance L, one complete wavelength such that :

$$\overline{(\quad)} = \int_0^L (\quad)dx .$$

Imagine a periodic function of $F(x)$ with wavelength L , such that F(L) = F(0)= $F(nL)$, where n is any integer. If we take the averaging operator of ∂F/∂x over distance L,

$$\overline{\partial F(x) / \partial x} = \int_0^L \frac{\partial F(x)}{\partial x}dx = F(L) - F(0) = 0$$

We multiply the upper- and lower-layer vorticity equations by ψ_1 and ψ_3 as a first step, considering the upper layer equation in detail, and then average them:

$$-\psi_1 \frac{\partial}{\partial t}\frac{\partial^2\psi_1}{\partial x^2} - U_1\psi_1 \frac{\partial}{\partial x}\frac{\partial^2\psi_1}{\partial x^2} - \beta\psi_1 \frac{\partial\psi_1}{\partial x} = -\frac{f_0}{\delta p}\psi_1\omega_2,$$
$$-\psi_1 \frac{\partial}{\partial x}\frac{\partial}{\partial t}\frac{\partial\psi_1}{\partial x} - U_1\psi_1 \frac{\partial}{\partial x}\frac{\partial^2\psi_1}{\partial x^2} - \beta\frac{1}{2}\frac{\partial\psi_1^2}{\partial x} = -\frac{f_0}{\delta p}\psi_1\omega_2,$$
$$-\frac{\partial}{\partial x}\left(\psi_1 \frac{\partial}{\partial t}\frac{\partial\psi_1}{\partial x}\right) + \frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial t}\frac{\partial\psi_1}{\partial x} - U_1\frac{\partial}{\partial x}\left(\psi_1 \frac{\partial}{\partial x}\frac{\partial\psi_1}{\partial x}\right) + U_1\frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial x}\frac{\partial\psi_1}{\partial x} - \beta\frac{1}{2}\frac{\partial\psi_1^2}{\partial x} = -\frac{f_0}{\delta p}\psi_1\omega_2,$$

Averaging by taking the $\overline{(\quad)}$ operator:

$$-\overline{\frac{\partial}{\partial x}\left(\psi_1 \frac{\partial}{\partial t}\frac{\partial\psi_1}{\partial x}\right)} + \overline{\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial\psi_1}{\partial x}\right)^2} - U_1\overline{\frac{\partial}{\partial x}\left(\psi_1 \frac{\partial}{\partial x}\frac{\partial\psi_1}{\partial x}\right)} + U_1\overline{\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\psi_1}{\partial x}\right)^2} - \beta\overline{\frac{1}{2}\frac{\partial\psi_1^2}{\partial x}} = -\frac{f_0}{\delta p}\overline{\psi_1\omega_2},$$

Since the first, third, fourth and fifth terms vanish because they are of the form $\overline{\partial F(x) / \partial x} = 0$, leaving:

$$\overline{\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial\psi_1}{\partial x}\right)^2} = -\frac{f_0}{\delta p}\overline{\psi_1\omega_2}$$

Thus, the time change of average kinetic energy upper layer $v_1^2 / 2$ is proportional to the correlation of pressure coordinate vertical velocity with upper layer streamfunction.

Similarly for the lower layer:

$$\overline{\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \psi_3}{\partial x}\right)^2} = \frac{f_0}{\delta p}\overline{\psi_3\omega_2}$$

Adding the kinetic energy equations together:

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\left[\left(\frac{\partial \psi_1}{\partial x}\right)^2 + \left(\frac{\partial \psi_3}{\partial x}\right)^2\right]} = -\frac{f_0}{\delta p}\overline{(\psi_1 - \psi_3)\omega_2}$$

This expression can be rewritten:

$$\frac{\partial}{\partial t}\frac{1}{2}(\overline{v_1^2} + \overline{v_3^2}) = -\frac{f_0}{\delta p}\overline{\left(\frac{\phi_1 - \overline{\phi}_1}{f_0} - \frac{\phi_3 - \overline{\phi}_2}{f_0}\right)\omega_2} = -\frac{1}{\delta p}\overline{[(\phi_1 - \phi_3) - (\overline{\phi}_1 - \overline{\phi}_3)\omega_2}$$
$$\approx \frac{1}{\delta p}\overline{R(T_2 - \overline{T}_2)\ln\frac{p_1}{p_3}\omega_2} = -\frac{R}{\delta p}\ln\frac{p_3}{p_1}\overline{(T_2 - \overline{T}_2)\omega_2} = -\frac{R\overline{T}}{\delta p}\ln\frac{p_3}{p_1}\frac{1}{\overline{\overline{T}}}\overline{(T_2 - \overline{T}_2)\omega_2}$$
$$= -\frac{\overline{\phi}_1 - \overline{\phi}_3}{\delta p}\frac{\overline{(T_2 - \overline{T}_2)\omega_2}}{\overline{T}}$$

Thus, if relatively cold air ($\delta T < 0$) is sinking ($\omega > 0$) and relatively warm air ($\delta T > 0$) is rising ($\omega < 0$) the correlation is negative and the kinetic energy is increasing. Conversely, if cold air is rising and warm air is sinking the correlation between ω and δT is positive and the kinetic energy is decreasing. The former case is called ***"Thermally Direct"*** motions and the latter is called ***"Thermally Indirect"*** motions.

Now we multiply the thermodynamic energy equation at level 2 by the thermal streamfunction,

$$(\psi_1 - \psi_3)\frac{\partial}{\partial t}(\psi_1 - \psi_3) + U_M(\psi_1 - \psi_3)\frac{\partial}{\partial x}(\psi_1 - \psi_3) - U_T(\psi_1 - \psi_3)\frac{\partial}{\partial x}(\psi_1 + \psi_3)$$
$$= \frac{\sigma\delta p}{f_0}(\psi_1 - \psi_3)\omega_2,$$
$$\frac{\partial}{\partial t}\tfrac{1}{2}(\psi_1 - \psi_3)^2 + U_M\frac{\partial}{\partial x}\tfrac{1}{2}(\psi_1 - \psi_3)^2 - U_T(\psi_1 - \psi_3)\frac{\partial}{\partial x}(\psi_1 + \psi_3) = \frac{\sigma\delta p}{f_0}(\psi_1 - \psi_3)\omega_2.$$

We take the average as before,

$$\frac{\partial}{\partial t}\overline{\tfrac{1}{2}(\psi_1 - \psi_3)^2} + U_M\frac{\partial}{\partial x}\overline{\tfrac{1}{2}(\psi_1 - \psi_3)^2} - U_T\overline{(\psi_1 - \psi_3)\frac{\partial}{\partial x}(\psi_1 + \psi_3)} = \frac{\sigma\delta p}{f_0}\overline{(\psi_1 - \psi_3)\omega_2},$$
$$\frac{\partial}{\partial t}\overline{\tfrac{1}{2}(\psi_1 - \psi_3)^2} + 0 - U_T\overline{(\psi_1 - \psi_3)\frac{\partial}{\partial x}(\psi_1 + \psi_3)} = \frac{\sigma\delta p}{f_0}\overline{(\psi_1 - \psi_3)\omega_2},$$

We now multiply through by the square of the inverse Rossby radius, $\lambda^2 = f_0^2 / \sigma(\delta p)^2$ as defined previously,

$$\frac{\partial}{\partial t}\frac{\lambda^2}{2}\overline{(\psi_1-\psi_3)^2}-U_T\frac{\lambda^2}{2}\overline{(\psi_1-\psi_3)\frac{\partial}{\partial x}(\psi_1+\psi_3)}=\frac{f_0}{\sigma\delta p}\overline{(\psi_1-\psi_3)\omega_2},$$

In more readily interpreted units, remembering the thermal wind equation

$$\frac{\partial}{\partial t}\frac{\lambda^2}{2}\left(\frac{R}{f_0}\ln\frac{p_3}{p_1}\right)^2\overline{(T_2-\overline{T}_2)^2}-U_T\frac{\lambda^2}{2}\frac{R}{f_0}\ln\frac{p_3}{p_1}\overline{(T_2-\overline{T}_2)v_2}=\frac{R}{\delta p}\ln\frac{p_3}{p_1}\overline{(T_2-\overline{T}_2)\omega_2}$$

$$\frac{\partial}{\partial t}\frac{\lambda^2}{2}\left(\frac{R\overline{T}_2}{f_0}\ln\frac{p_3}{p_1}\right)^2\overline{\left(\frac{T_2-\overline{T}_2}{\overline{T}_2}\right)^2}+\left(\frac{R}{f_0}\frac{\partial\overline{T}_2}{\partial y}\ln\frac{p_3}{p_1}\right)\frac{\lambda^2}{2}\frac{R\overline{T}_2}{f_0}\ln\frac{p_3}{p_1}\overline{\frac{T_2-\overline{T}_2}{\overline{T}_2}v_2}=\frac{R\overline{T}_2}{\delta p}\ln\frac{p_3}{p_1}\overline{\frac{T_2-\overline{T}_2}{\overline{T}_2}\omega_2}$$

$$\frac{\lambda^2}{2}\left(\frac{\overline{\phi}_1-\overline{\phi}_3}{f_0}\right)^2\frac{\partial}{\partial t}\overline{\left(\frac{T_2-\overline{T}_2}{\overline{T}_2}\right)^2}=-\frac{\lambda^2}{2}\left(\frac{\overline{\phi}_1-\overline{\phi}_3}{f_0}\right)^2\frac{1}{\overline{T}_2}\frac{\partial\overline{T}_2}{\partial y}\overline{\frac{T_2-\overline{T}_2}{\overline{T}_2}v_2}+\frac{\overline{\phi}_1-\overline{\phi}_3}{\delta p}\overline{\frac{T_2-\overline{T}_2}{\overline{T}_2}\omega_2}$$

Let’s take a closer look at the inverse Rossby radius squared,

$$\lambda^2=\frac{f_0^2}{\frac{\partial\overline{\phi}}{\partial p}\frac{1}{\overline{\theta}}\frac{\partial\overline{\theta}}{\partial p}(\delta p)^2}\approx\frac{f_0^2\overline{\theta}}{(\overline{\phi}_1-\overline{\phi}_3)(\overline{\theta}_1-\overline{\theta}_3)}$$

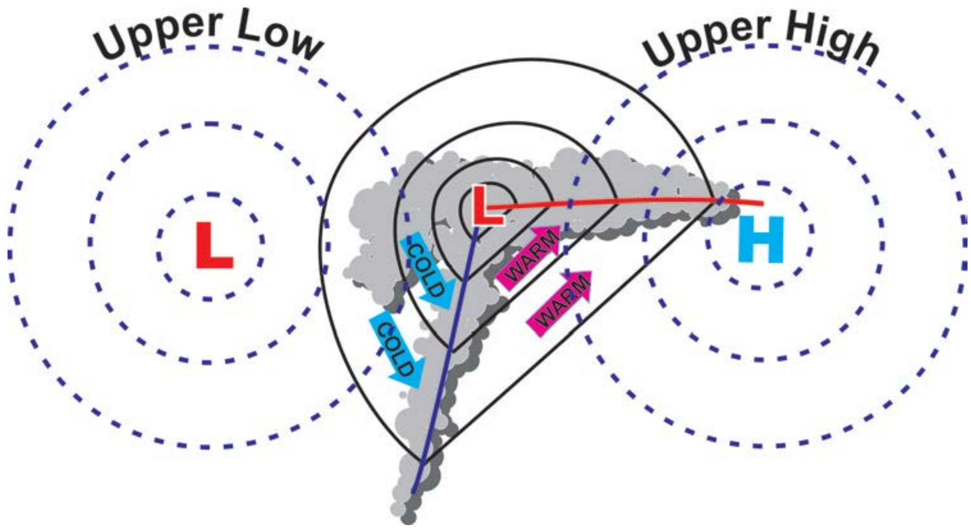
Substituting into the potential energy equation,

$$\frac{1}{2}\frac{\overline{\theta}_2(\overline{\phi}_1-\overline{\phi}_3)}{\overline{\theta}_1-\overline{\theta}_3}\frac{\partial}{\partial t}\overline{\left(\frac{T_2-\overline{T}_2}{\overline{T}_2}\right)^2}=-\frac{1}{2}\frac{\overline{\theta}_2(\overline{\phi}_1-\overline{\phi}_3)}{\overline{\theta}_1-\overline{\theta}_3}\frac{1}{\overline{T}_2}\frac{\partial\overline{T}_2}{\partial y}\overline{\frac{T_2-\overline{T}_2}{\overline{T}_2}v_2}+\frac{\overline{\phi}_1-\overline{\phi}_3}{\delta p}\overline{\frac{T_2-\overline{T}_2}{\overline{T}_2}\omega_2}$$

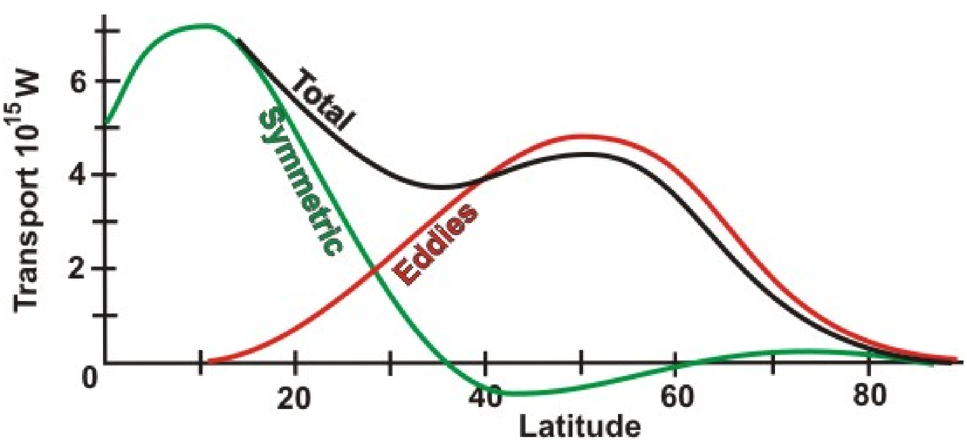
The first term is proportional to the integrated squares of the temperature deviations at the middle level. This quantity is the available potential energy of the baroclinicly unstable waves. The second term is the product of the meridional temperature gradient times the meridional eddy flux of temperature. Since the meridional temperature gradient is negative, when cold air ($\delta T < 0$) is moving southward ($v < 0$) and warm air ($\delta T > 0$) is moving northward, the term (including the leading minus sign) is positive and the available potential energy increases. If either the temperature gradient or the eddy temperature flux (but not both) reverse, the available potential energy will decrease with time. The essence of this process is a down-gradient eddy heat flux from warm to cold air. The last term is the conversion of available potential energy into kinetic energy. Note that here the conversion term has opposite sign from the kinetic energy equation. If we add the two energy equations together, the last term cancels,

$$\frac{\partial}{\partial t}\frac{1}{2}(\overline{v_1^2}+\overline{v_3^2})+\frac{\overline{\theta}_2(\overline{\phi}_1-\overline{\phi}_3)}{\overline{\theta}_1-\overline{\theta}_3}\frac{\partial}{\partial t}\overline{\frac{1}{2}\left(\frac{T_2-\overline{T}_2}{\overline{T}_2}\right)^2}=-\frac{1}{2}\frac{\overline{\theta}_2(\overline{\phi}_1-\overline{\phi}_3)}{\overline{\theta}_1-\overline{\theta}_3}\frac{1}{\overline{T}_2}\frac{\partial\overline{T}_2}{\partial y}\overline{\frac{T_2-\overline{T}_2}{\overline{T}_2}v_2}+0$$

The function of unstable baroclinic waves is to move heat and westerly momentum poleward. As shown below, southerly meridional flow, more westerly zonal flow, and warmer temperatures correlate with each other; as do northerly meridional flow, less westerly zonal flow, and colder temperatures. Warm air moving northward ascends and cold air moving southward sinks.



In the larger scheme of things baroclinic eddies accomplish the transport in middle latitudes; whereas the mean meridional circulation does the job in the tropics and to a lesser extent in the Arctic and Antarctic. The reasons that we don't see unstable baroclinic waves in the tropics are because the temperature gradients are fairly weak, and because the beta effect suppresses Baroclinic instability. Recall that $\beta = (2\Omega/R_E)\cos\varphi$.



Tropospheric Energy Cycle:

It turns out that the middle latitude Farrell Cell is thermally indirect so that the energy balance of the middle latitude atmosphere follows the now-famous ***Lorenz Cycle***