Objective: Gain understanding of forced symmetric circulations & barotropic instability

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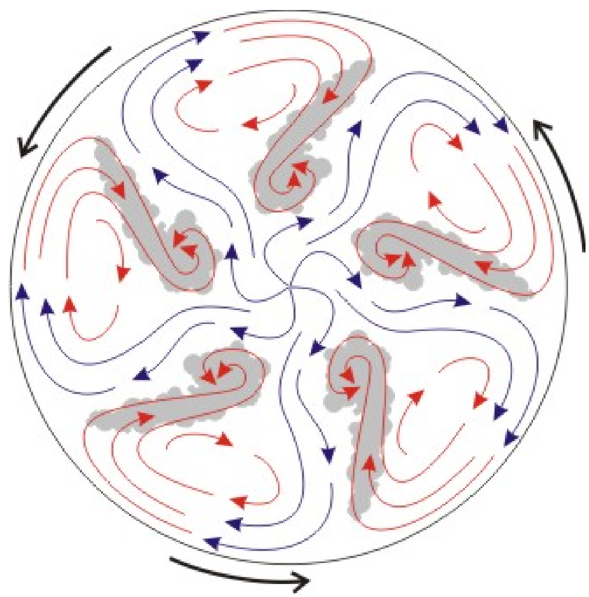
Reading: Holton pp. 277-279, 506-507 (not very good)

Topics:

- Overview
- Meridional structure
- Forced meridional circulations
- Barotropic instability

Overview:

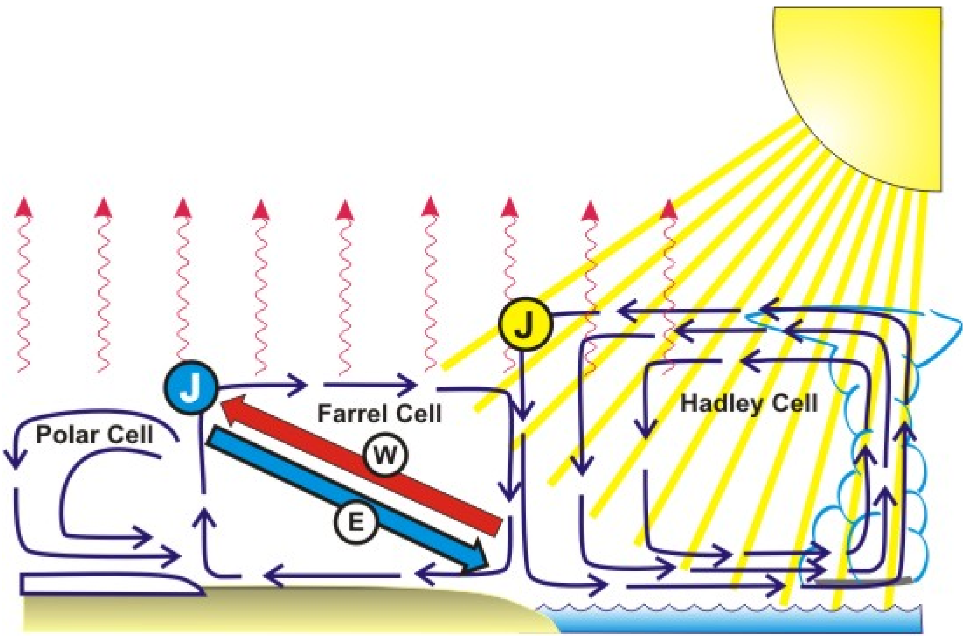
View from over the North Pole:



- Easterlies in low latitudes. where (in a fixed reference frame) the air moves more slowly than the surface
- Westerlies in high latitudes, where the air moves faster than the surface.
- The Westerlies are dominated by irregular eddies, called Middle Latitude Cyclones:
  - In contrast with the steady Easterlies
  - Eddies lean eastward with latitude
  - Warm air with more westerly momentum moves poleward and upward
  - Cool air with less westerly momentum moves equatorward and downward.
  - Resulting in poleward eddy transport of heat and westerly (i.e., with the planet's rotation) angular momentum.
- Friction with the surface increases westerly momentum in the tropics [where the air (in a fixed reference frame) rotates more slowly than the Earth's surface] and decreases westerly momentum in middle latitudes (where the air rotates more rapidly than the surface)
- Excess of solar heating (insolation) over infrared radiation to space in the tropics warms the sea and air; whereas a deficit of insolation below infrared radiation in middle and high latitudes cools the sea and air.

- Atmospheric (and oceanic) circulations redistribute heat and angular momentum meridionally to maintain balance against these forcings.

Mean Meridional Structure:



Recall that:

- The Sun heats the tropics and surface friction in the Trade Winds adds westerly momentum
- The Hadley cell exports heat and westerly momentum to middle latitudes
- It is thermally direct inasmuch as warmer air rises in the ITCZ and cooler air sinks in the subtropics.
- In Middle Latitudes quasi-horizontal eddies take over the transport
  - Warmer air with excess westerly momentum moves poleward and ascends
  - Cooler air with a deficit of westerly momentum moves equatorward and sinks.
  - The Ferrel Cell is forced by these eddies and by friction. It is thermally indirect: cooler air rises and warmer air sinks.

Dynamics of Forced Meridional Circulations:

Start with the Zonal Momentum, Thermodynamic Energy, and Continuity equations,

$$\frac{\partial u}{\partial t} - v\left(f - \frac{\partial u}{\partial y}\right) + \omega\frac{\partial u}{\partial p} = F$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial p}\right) + \sigma\omega = -q$$

$$\frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

The zonal wind is in geostrophic balance,

$$fu = -\frac{\partial \phi}{\partial y}.$$

So that,

$$f\frac{\partial u}{\partial t} = -\frac{\partial^2 \phi}{\partial y \partial t}, \text{ and } f\frac{\partial u}{\partial p} = -\frac{\partial^2 \phi}{\partial y \partial p}$$

Multiply the zonal momentum equation by f ,

$$f\frac{\partial u}{\partial t} - f\left(f - \frac{\partial u}{\partial y}\right)v + f\frac{\partial u}{\partial p}\omega = fF \, .$$

Substituting for ∂u/∂t from geostrophic balance and writing $S = \partial u/\partial p$,

$$-\frac{\partial^2 \phi}{\partial y \partial t} - f\left(f - \frac{\partial u}{\partial y}\right)v + fS\omega = fF$$

Since $fS = -\partial^2\phi/\partial p\partial y$, the thermodynamic energy equation becomes,

$$\frac{\partial^2 \phi}{\partial p \partial t} - fSv + \sigma\omega = -q$$

Taking ∂/∂p of the momentum equation and ∂/∂y of the thermodynamic equation, assuming (for simplicity) that f σ, S, and $\partial u/\partial y$, but not F and q, are spatially constant.

$$-\frac{\partial^3 \phi}{\partial p \partial t \partial y} - f\left(f - \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial p} + fS\frac{\partial \omega}{\partial p} = \frac{\partial (fF)}{\partial p},$$

$$\frac{\partial^3 \phi}{\partial p \partial t \partial y} - fS\frac{\partial v}{\partial y} + \sigma\frac{\partial \omega}{\partial y} = -\frac{\partial q}{\partial y}$$

Adding,

$$0 - f\left(f - \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial p} + fS\frac{\partial \omega}{\partial p} - fS\frac{\partial v}{\partial y} + \sigma\frac{\partial \omega}{\partial y} = \frac{\partial (fF)}{\partial p} - \frac{\partial q}{\partial y}.$$

Introduce a streamfunction, such that,

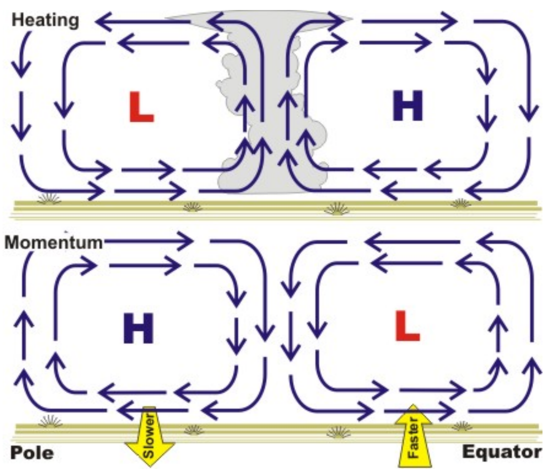
$$\omega = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial p}$$

Substitute,

$$f\left(f-\frac{\partial u}{\partial y}\right)\frac{\partial^2\psi}{\partial p^2}+fS\frac{\partial^2\psi}{\partial y\partial p}+fS\frac{\partial\psi}{\partial y\partial p}+\sigma\frac{\partial^2\omega}{\partial y^2}=\frac{\partial(fF)}{\partial p}-\frac{\partial q}{\partial y}$$

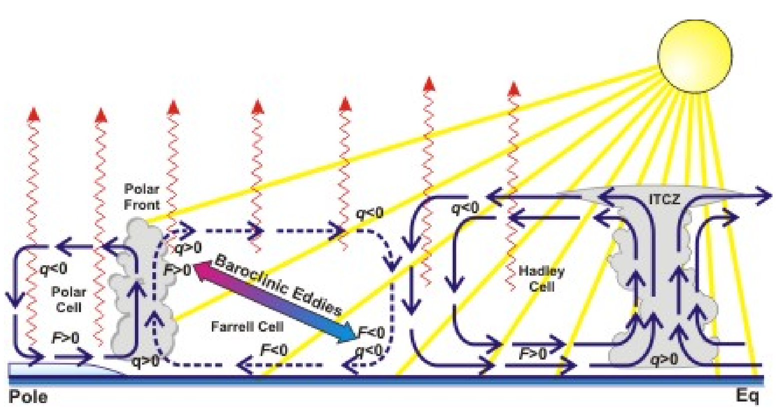
Collect terms,

$$f\left(f-\frac{\partial u}{\partial y}\right)\frac{\partial^2\psi}{\partial p^2}+2fS\frac{\partial^2\psi}{\partial y\partial p}+\sigma\frac{\partial^2\omega}{\partial y^2}=\frac{\partial(fF)}{\partial p}-\frac{\partial q}{\partial y}$$



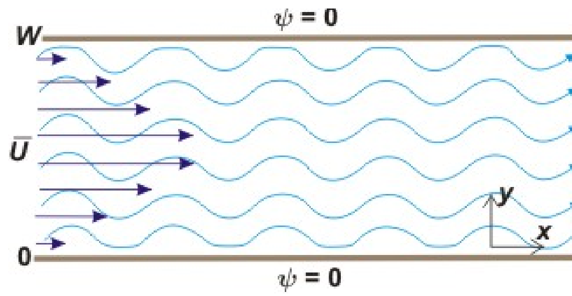
This is a form of the Sawyer-Eliassen equation. Mathematically, it is Poisson equation provided that the coefficients of the unmixed second derivatives are positive and that the coefficient of the mixed partial is not too large. It is an idealization since we have neglected significant spatial variations of the coefficients. Note that the forcing is proportional to the horizontal derivative of imposed heating and the vertical derivative of the imposed momentum sources or sinks. Actual forcing mechanisms include convective heating, radiative cooling (or heating), surface friction, and eddy sources or sinks of heat and momentum due to synoptic scale weather systems.

Imposed heating forces a streamfunction dipole with ascending motion through the heat source fed by low-level horizontal convergence. Above the heat source horizontal divergence carries air outward throughout a region of gradual descent, whose horizontal scale is determined by the Rossby radius of deformation. Imposed angular momentum sources force horizontal flow away from the axis of rotation and toward the equator; whereas imposed angular momentum sinks force horizontal flow toward from the axis of rotation and away from the equator. Combinations of these forcings cause the classic three-cell model of the meridional secondary circulation.



Barotropic Instability:

Purely barotropic (i.e., no vertical shear due to horizontal temperature gradients), nondivergent flow conserves vorticity. Here we linearize the vorticity equation, allowing $\overline{u}$ to be a function of *y*.



$$\left(\frac{\partial}{\partial t}+\overline{u}(y)\frac{\partial}{\partial x}\right)\zeta'+v'\frac{\partial(\overline{\zeta}+f)}{\partial y}=0.$$

Introduce a streamfunction, ψ, such that,

$$u'=-\frac{\partial\psi}{\partial y},v'=\frac{\partial\psi}{\partial x}$$

$$\left(\frac{\partial}{\partial t}+\overline{u}(y)\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\frac{\partial\psi}{\partial x}\frac{\partial(\overline{\zeta}+f)}{\partial y}=0.$$

Assume wavelike solutions with meridionally varying amplitude,

$$\psi(x,y,t)=\mathrm{Re}\{\Psi_0(y)\exp ik(x-ct)\}$$

Recall that $c = c_R + ic_I$ and $\Psi_0 = \Psi_R + i\Psi_I$ may be complex.

$$ik[\overline{u}(y)-c]\left(\frac{d^2\Psi_0}{dy^2}-k^2\Psi_0\right)+ik\Psi_0\frac{\partial(\overline{\zeta}+f)}{\partial y}=0$$

The wave is confined to a channel of width *W*, such that $\psi(0) = 0$ and $\psi(W) = 0$. Multiply through by $\Psi_0^* = \Psi_R - \Psi_I$, divide by $\overline{u}-(c_R+ic_I)$,

$$\Psi_0^*\frac{d^2\Psi_0}{dy^2}-k^2\Psi_0^*\Psi_0+\frac{\Psi_0^*\Psi_0}{\overline{u}-c}\frac{\partial(\overline{\zeta}+f)}{\partial y}=0\,.$$

Recall that $\Psi_0^*\Psi_0=|\Psi_0|^2$ and

$$\frac{1}{\overline{u}-(c_R+ic_I)}=\frac{1}{(\overline{u}-c_R)-ic_I}=\frac{\overline{u}-c_R+ic_I)}{(\overline{u}-c_R)^2+c_I^2}=\frac{\overline{u}-c_R+ic_I}{|\overline{u}-c|^2}$$

Integrate from 0 to W. The imaginary part of the integral becomes (note sign switch):

$$\int_0^W\left[\Psi_I\frac{d^2\Psi_R}{dy^2}-\Psi_R\frac{d^2\Psi_I}{dy^2}\right]dy=c_I\int_0^W\frac{\partial(\zeta+f)}{\partial y}\frac{|\Psi_0|^2}{|\overline{u}-c|^2}dy$$

If we add and subtract $(d\Psi_I/dy)(d\Psi_R/dy)$, the left side integrates to:

$$\int_0^W \left[\Psi_I \frac{d^2\Psi_R}{dy^2} + \frac{d\Psi_I}{dy}\frac{d\Psi_R}{dy} - \left(\frac{d\Psi_I}{dy}\frac{d\Psi_R}{dy} + \Psi_R \frac{d^2\Psi_I}{dy^2} \right) \right] dy$$

$$= \int_0^W \frac{d}{dy}\left[\Psi_I \frac{d\Psi_R}{dy} - \Psi_R \frac{d\Psi_I}{dy} \right] dy = \left[\Psi_I \frac{d\Psi_R}{dy} - \Psi_R \frac{d\Psi_I}{dy} \right]_0^W = 0$$

Thus, the left side must be identically zero. The right side must also be zero. This condition can come about if $c_I = 0$ when the waves are stable or if the integral on the right is zero:

$$\int_0^W \frac{\partial(\zeta + f)}{\partial y} \frac{|\Psi_0|^2}{|\overline{u} - c|^2} dy = 0$$

Since $|\Psi_0|^2 > 0$ and $|\overline{u} - c|^2 > 0$, the only way for c_I to be nonzero is for $\partial(\overline{\zeta} + f)/\partial y$ to change sign in such a way that the integral of the vorticity gradient weighted by the ratio of the amplitude of the streamfunction squared to the square of the amplitude of the complex phase speed is zero. In that situation instability (nonzero c_I) is possible. Thus reversal of the sign of the vorticity gradient is a ***Necessary, but not a Sufficient, condition for Barotropic Instability***.