Objective: Gain understanding of shallow-water waves on and equatorial beta plane

----

Reading: Holton pp. 394-400

Topics:

- Introduction
- Governing equations
- General solutions
    - Rossby waves
    - Inertia-gravity waves
- A special case—Kelvin waves
- Summary

Reference:

Matsuno, T., 1966: Quasi-geostrophic motions in the equatorial area. *J. Metor. Soc. Japan*, **44**(1), 25-42.

Introduction:

We have spent a lot of effort on the middle latitude beta-plane where the change of the Coriolis parameter across the domain is small compared with the value of the Coriolis parameter itself. Since the Coriolis parameter, $f = 2\Omega \sin\varphi$ decreases toward the equator, the Rossby parameter $\beta = \partial f / \partial y = 2\Omega s \cos\varphi / A$ increases toward the equator. Here Ω is the angular velocity of the Earth's rotation, φ is latitude, and A is the radius of the earth, such that in radians, $\varphi = y / A$. Near the equator it is convenient to define *y* as the great-circle distance north or south of the equator. We represent the Coriolis parameter as $f = \beta y$. The geostrophic wind relation for the zonal wind is,

$$u_g = -\frac{1}{f}\frac{\partial \phi}{\partial y}.$$

Contrary to what one might think, the geostrophic wind can be both nonzero and finite at the equator when both the Coriolis parameter and the pressure gradient go to zero. From L'Hospital's rule for a zero-over-zero situation,

$$u_g = -\frac{\partial / \partial y(\partial \phi / \partial y)}{\partial f / \partial y} = -\frac{\partial^2 \phi / \partial y^2}{\beta}$$

Since the pressure gradient is zero ϕ must have a maximum, $\partial^2\phi/\partial y^2 < 0$, a minimum $\partial^2\phi/\partial y^2 > 0$, or an inflection point, $\partial^2\phi/\partial y^2 = 0$, at the equator where $\partial\phi/\partial y = 0$ and $f = 0$ as well. Sometimes the wind-pressure balance is more complicated.

In this formulation, the linearized, shallow-water zonal momentum, meridional momentum, and mass continutiy equations are:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \beta y v = -\frac{\partial \phi}{\partial x},\\
&\frac{\partial v}{\partial t} + \beta y u = -\frac{\partial \phi}{\partial y},\\
&\frac{\partial \phi}{\partial t} = -C^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).
\end{aligned}$$

Here (u, v) are the zonal and meridional velocity components $\phi = gh$ is the geopotential, $C^2 = gH$ is the gravity-wave phase speed, and $f = \beta y$ as already explained. Making the “Magic Operators” as before:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \beta y \frac{\partial v}{\partial t} = -\frac{\partial^2 \phi}{\partial x \partial t},\\
&\frac{\partial^2 v}{\partial t^2} + \beta y \frac{\partial u}{\partial t} = -\frac{\partial^2 \phi}{\partial y \partial t},\\
&\frac{\partial^2 u}{\partial t^2} - \beta y\left(-\frac{\partial \phi}{\partial y} - \beta y u\right) = -\frac{\partial^2 \phi}{\partial x \partial t},\\
&\frac{\partial^2 v}{\partial t^2} - \beta y\left(-\frac{\partial \phi}{\partial x} + \beta y v\right) = -\frac{\partial^2 \phi}{\partial y \partial t},\\
&\left[\frac{\partial^2}{\partial t^2} + (\beta y)^2\right] u = -\frac{\partial^2 \phi}{\partial x \partial t} - \beta y \frac{\partial \phi}{\partial y},\\
&\left[\frac{\partial^2}{\partial t^2} + (\beta y)^2\right] v = +\beta y \frac{\partial \phi}{\partial x} - \frac{\partial^2 \phi}{\partial y \partial t},
\end{aligned}$$

Substituting into the continuity equation:

$$\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2} + (\beta y)^2\right]\phi + C^2\left[\frac{\partial}{\partial x}\left(-\frac{\partial^2 \phi}{\partial x \partial t} - \beta y \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\beta y \frac{\partial \phi}{\partial x} - \frac{\partial^2 \phi}{\partial y \partial t}\right) - 2\beta^2 y v\right] = 0$$

Neglecting the last term inside the square brackets before the equal sign, expanding the partial derivatives, and simplifying:

$$\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2}+(\beta y)^2\right]\phi+C^2\left[-\frac{\partial^2\phi}{\partial t\partial x^2}-\beta y\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial t\partial y^2}+\beta y\frac{\partial^2\phi}{\partial x\partial y}+\beta\frac{\partial\phi}{\partial x}\right]=0,$$
$$\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2}+(\beta y)^2\right]\phi+C^2\left[-\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\beta\frac{\partial\phi}{\partial x}\right]=0.$$

Assume solutions of the form $\phi=\Phi(y)\exp i(\omega t-kx)$ that we <u>require</u> to be bounded as y becomes large,

$$i\omega[-\omega^2+(\beta y)^2]\Phi+C^2\left[i\omega\left(k^2-\frac{d^2\Phi}{dy^2}\right)-ik\beta\Phi\right]=0.$$

Rearranging,

$$i\omega[-\omega^2+(\beta y)^2]\Phi+C^2\left[i\omega\left(k^2-\frac{d^2\Phi}{dy^2}\right)-ik\beta\Phi\right]=0,$$
$$\frac{d^2\Phi}{dy^2}+\left(\frac{\omega^2}{C^2}-k^2+\frac{k\beta}{\omega}-\frac{(\beta y)^2}{C^2}\right)\Phi=0.$$

This leads us to the "Meridional Structure Equation (MSE)," which we will solve analytically for $\Phi(y)$ given the parameters of the problem, C^2, ω, and β. Note that all of the quantities in parentheses have dimensions of reciprocal length. We can combine them to produce scaling factors with units $\mathrm{m}^{-2}=\beta/C$ and $\mathrm{s}^{-2}=\beta C$. Factoring out β/C from the above equation yields,

$$\frac{d^2\Phi}{dy^2}+\frac{\beta}{C}\left(\frac{\omega^2}{\beta C}-\frac{C}{\beta}k^2+\frac{kC}{\omega}-\frac{\beta y^2}{C}\right)\Phi=0,$$
$$\frac{C}{\beta}\frac{d^2\Phi}{dy^2}+\left(\frac{\omega^2}{\beta C}-\frac{C}{\beta}k^2+\frac{k(\beta C)^{1/2}(C/\beta)^{1/2}}{\omega}-\frac{\beta y^2}{C}\right)\Phi=0$$

If we define $y'=(\beta/C)^{1/2}y$, $k'=(C/\beta)^{1/2}k$, and $\omega'=\omega/(\beta C)^{1/2}$, the MSE becomes in nondimensional form:

$$\frac{d^2\Phi}{dy'^2}+\left(\omega'^2-k'^2+\frac{k'}{\omega'}-y'^2\right)\Phi=0$$

We write $\Phi(y')=F(y')\exp(-y'^2/2)$ such that $d\Phi/dy' = [dF/dy' - y'F]\exp\{-y'^2/2\}$ and $d^2\Phi/dy'^2 = [d^2F/dy'^2 - 2y'dF/dy' + y'^2F - F]\exp\{-y^2/2\}$ the MSE (dropping primes for simplicity) is transformed into:

$$\left(\frac{d^2F}{dy^2}-2y\frac{dF}{dy}+y^2F-F\right)e^{-y^2/2}+\left(\omega^2-k^2+\frac{k}{\omega}-y^2\right)Fe^{-y^2/2}=0.$$

Adding the explicit Gaussian to the solution is a big help in keeping the solution bounded for large y, but not, as we shall see, a panacea. The equation simplifies to,

$$\frac{d^2F}{dy^2}-2y\frac{dF}{dy}+\left(\omega^2-k^2+\frac{k}{\omega}-1\right)F$$
$$\frac{d^2F}{dy^2}-2y\frac{dF}{dy}+2\lambda F=0$$

This equation is known as Hermite's Equation. It solutions are well known because it plays an important role in Quantum Mechanics, where it yields the eigenvalues of the classical QM harmonic oscillator. We seek power-series solutions:

$$F(y)=\sum_{n=0}^{\infty}a_n y^n$$

Substituting into the much-manipulated MSE,

$$\sum_{n=2}^{\infty}n(n-1)a_n y^{n-2}-\sum_{n=1}^{\infty}2na_n y^n+\sum_{n=0}^{\infty}2\lambda a_n y^n=0$$

We shift the summation of the first summation to start at *n* = 0,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}y^n-\sum_{n=1}^{\infty}2na_n y^n+\sum_{n=0}^{\infty}2\lambda a_n y^n=0$$

The second summation starts at 1, but we notice that the zeroth (n = 0) term is zero so that we may write,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}y^n-\sum_{n=0}^{\infty}2na_n y^n+\sum_{n=0}^{\infty}2\lambda a_n y^n=0$$

Since the coefficients of each power of *y* must add up to zero individually, the relation between a_{n+2} and a_n is:

$$a_{n+2}=\frac{2(n-\lambda)}{(n+2)(n+1)}a_n$$

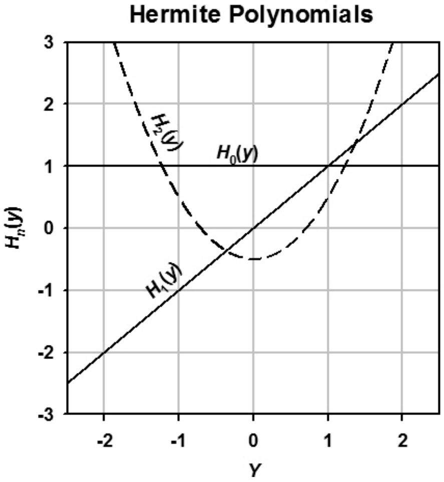
Both a_0 and a_1 are arbitrary so that there are two distinct series for any value of λ. Let's try a whole number, λ = 5. The first (i.e. n =2) term starting with a_0 is:

$$
\begin{aligned}
a_2 &= \frac{2(0-5)}{(0+2)(0+1)}a_0 = \frac{-10}{2}a_0 = 5a_0 \\
a_4 &= \frac{2(2-5)}{(2+2)(2+1)}a_2 = \frac{-6}{12}a_2 = -\frac{1}{2}a_2 = -\frac{5}{2}a_0 \\
a_6 &= \frac{2(4-5)}{(4+2)(4+1)}a_4 = \frac{-6}{(6)(5)}a_4 = -\frac{1}{5}a_4 = \frac{1}{2}a_0 \\
a_8 &= \frac{2(6-5)}{(6+2)(6+1)}a_6 = \frac{2}{(8)(7)}a_6 = \frac{2}{56}a_6 = \frac{1}{56}a_0 \\
a_{10} &= \ldots
\end{aligned}
$$

For any odd integer value of λ or any non-integer value of λ, the series starting with a_0 will have an infinite number of terms. Successive terms will also be larger than the corresponding terms in the power series representation of exp($y^2/2$). Take for example the 8th term of the exponential $b_6 = b_0/6! = b_0/720$. What this tells us is that this infinite series solution times exp($-y^2/2$) will not be bounded for large *y*. Let's try the same analysis starting with a_1.

$$
\begin{aligned}
a_3 &= \frac{2(1-5)}{(1+2)(1+1)}a_1 = \frac{-8}{(3)(2)}a_1 = -\frac{4}{3}a_1, \\
a_5 &= \frac{2(3-5)}{(3+2)(3+1)}a_1 = \frac{-4}{(5)(4)}a_1 = -\frac{1}{5}a_1 = \frac{4}{15}a_0, \\
a_7 &= \frac{2(5-5)}{(5+2)(5+1)}a_1 = 0.
\end{aligned}
$$

More generally λ has to be an integer or else the power series will be infinite and so will its product with exp($-y^2/2$). If λ is even, the series starting with a_0 will have a finite number of terms; if λ is odd the series starting with a_1 will have a finite number terms; and both will be bounded when multiplied them by exp($-y^2/2$). All other solutions (i.e., λ ≠ 0, 1, 2, 3, 4, …) will be unbounded for large *y*. The solution that we obtained for λ = 5 was:



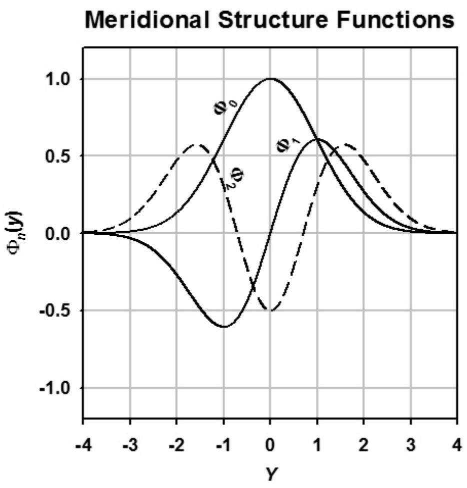
$$F_5 = a_1\left(x - \frac{4}{3}x^3 + \frac{4}{15}x^5\right)$$

Since a_1 is arbitrary, we may rescale this result as,

$$H_5 = 32x^5 - 160x^3 + 120x$$

These solutions, one for each value of λ, are called (physicist's) Hermite polynomials. Scaled in the conventional way, the first few Hermite polynomials are:

$$\begin{aligned} H_0 &= 1, \\ H_1 &= 2y, \\ H_2 &= 4x^2 - 2, \\ H_3 &= 8x^3 - 12x, \\ H_4 &= 16x^4 - 48x^2 + 12. \end{aligned}$$

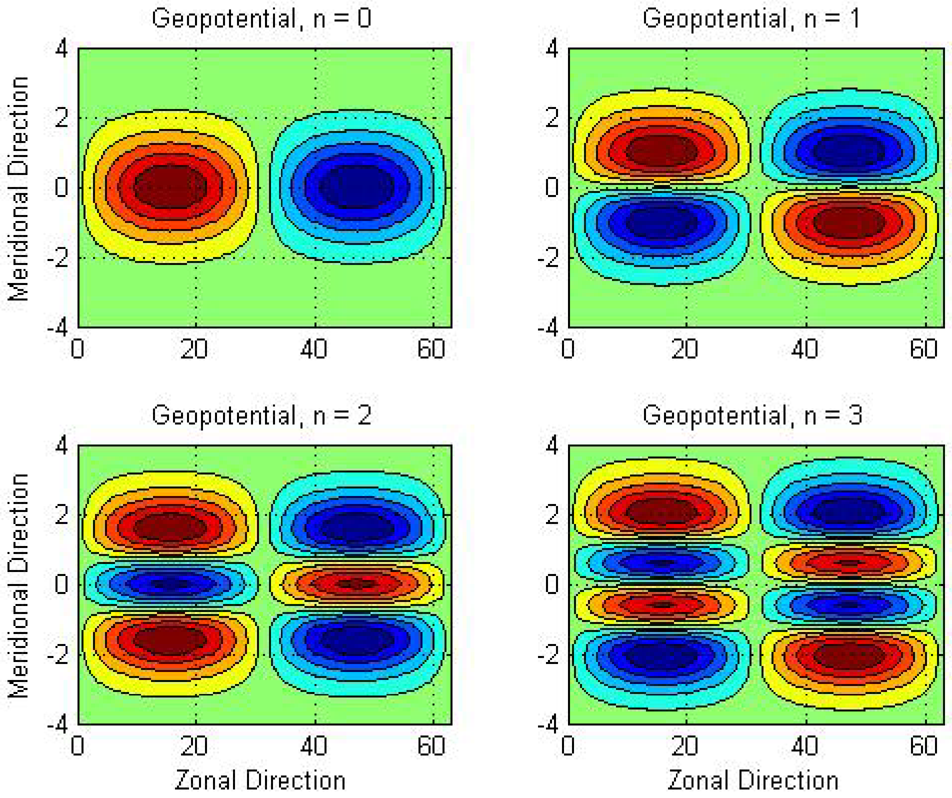


Since λ has to be an integer, its value imposes a condition on the frequency and zonal wavenumber. Remember when we introduced λ into the MSE, we set

$$\left(\omega^2 - k^2 - \frac{k}{\omega} - 1\right) = 2\lambda$$

This expression rearranges to,

$$\omega(\omega^2 - k^2) - (2\lambda + 1)\omega - k = 0$$



As in the homework, we separate high and low-frequency families of solutions,

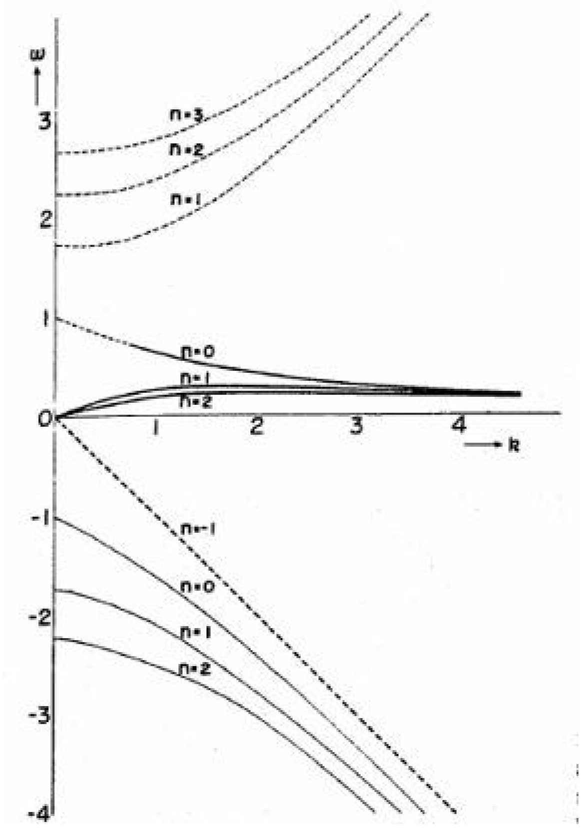
$$\begin{aligned} &\omega^2 - k^2 - (2\lambda + 1) = 0, \\ &\omega(-k^2) - (2\lambda + 1)\omega - k = 0. \end{aligned}$$

Putting the primes back in and solving for ω′,

$$\omega' = \pm\sqrt{k'^2 + (2\lambda+1)} = 0,$$
$$\omega' = -\frac{k'}{k'^2 + 2\lambda + 1}.$$

Restoring the equations to dimensional form:

$$\frac{\omega}{(\beta C)^{1/2}} = \pm\sqrt{k^2\frac{C}{\beta} + (2\lambda+1)} = \pm\frac{C^{1/2}}{\beta^{1/2}}\sqrt{k^2 + \frac{\beta}{C}(2\lambda+1)},$$
$$\frac{\omega}{(\beta C)^{1/2}} = -\frac{k\left(\dfrac{C}{\beta}\right)^{1/2}}{k^2\dfrac{C}{\beta} + 2\lambda + 1} = -\frac{k\left(\dfrac{\beta}{C}\right)^{1/2}}{k^2 + \dfrac{\beta}{C}(2\lambda+1)}.$$



And simplifying,

$$\omega = \pm C\sqrt{k^2 + \frac{\beta}{C}(2\lambda+1)},$$
$$\omega = -\frac{\beta k}{k^2 + \dfrac{\beta}{C}(2\lambda+1)}.$$

Here β / C has the units of inverse length squared so that it is effectively the equatorial inverse Rossby radius. The first two solutions are clearly rotational shallow-water gravity waves. In the limit of large *k*, their frequency approaches ±*kC*, the shallow-water gravity wave frequency; in the limit of small k, the frequency approaches $[\beta C(2\lambda+1)]^{1/2}$, which is a sort of inertia frequency. The final solution represents a divergent, two dimensional, Rossby wave. Note that for each value of λ = 0, 1, 2, 3, …, there is a (rapidly) westward propagating gravity wave, a (rapidly) eastward propagating gravity wave, and a (slowly) westward propagating Rossby wave, all of which share the same Hermite polynomial meridional structure. In this respect, these solutions differ from Matsuno's because we have solved for the geopotential rather than for the meridional velocity. His solutions are linear combinations of the ones presented here.

The solutions for λ = 0 require special treatment. Starting with the nondimensional characteristic equation,

$$\omega'(\omega'^2 - k'^2) - \omega' - k' = \omega'(\omega' - k')(\omega' + k') - \omega' - k'$$
$$(\omega' + k')[\omega'(\omega' - k') - 1] = (\omega' - k')(\omega'^2 - \omega' k' - 1) = 0$$

Setting each factor to zero and solving the quadratic equation for ω in the second factor,

$$(\omega' + k') = 0 \quad \Rightarrow \quad \omega' = -k'$$
$$(\omega'^2 - \omega' k' - 1) = 0 \quad \Rightarrow \quad \omega' = +\frac{k'}{2} \pm \sqrt{\frac{k'^2}{4} + 1}$$

Redimensionalizing,

$$\frac{\omega}{(\beta C)^{1/2}} = -k\left(\frac{C}{\beta}\right)^{1/2} \quad \Rightarrow \quad \omega = -Ck$$
$$\frac{\omega}{(\beta C)^{1/2}} = \frac{k}{2}\left(\frac{C}{\beta}\right)^{1/2} \pm \sqrt{\frac{k^2}{4}\frac{C}{\beta} + 1} \Rightarrow \omega = C\frac{k}{2} \pm \sqrt{C^2\frac{k^2}{4} + \beta C}$$

The first root represents a one-dimensional gravity wave. Separating the ± roots, the quadratic simplifies to,

$$\omega = C\left[\frac{k}{2} + \sqrt{\frac{k^2}{4} + \frac{\beta}{C}}\right] \quad \text{and}\, C\left[\frac{k}{2} - \sqrt{\frac{k^2}{4} + \frac{\beta}{C}}\right]$$

The first root is simply rotational gravity waves, as we have seen before. Consider the second root when *k* is large,

$$\omega = C\left[\frac{k}{2} - \frac{k}{2}\sqrt{1 + \frac{4\beta}{Ck^2}}\right] \approx C\left[\frac{k}{2} - \frac{k}{2}\left(1 + \frac{1}{2}\frac{4\beta}{Ck^2}\right)\right] = -\frac{\beta}{k},$$

Which expression describes a one dimensional Rossby wave. On the other hand, in the limit that *k* approaches zero,

$$\omega = -\left(\beta C\right)^{1/2},$$

just as the inertia-gravity waves do. Thus the n = 0 wave is a combined Rossby-(inertia) gravity wave.

A final special case corresponds to λ = −1. If we were to go back to the polarization relations, λ = 0 implies that the meridional velocity is always zero. This is an example of a Kelvin wave, which was first discovered in the context of waves propagating parallel with the coast in shallow-water. The governing equations with v = 0 are:

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x},$$
$$\frac{\partial \phi}{\partial y} = -\beta y u,$$
$$\frac{\partial \phi}{\partial t} = -C^2 \frac{\partial u}{\partial x}.$$

Differentiating the first expression with respect to time and substituting from the last,

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 \phi}{\partial t \partial x} = -\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial x}\left(-C^2 \frac{\partial u}{\partial x}\right) = C^2 \frac{\partial^2 u}{\partial x^2},$$

So the waves propagate with the gravity wave speed $C = \sqrt{gH}$, even though the zonal wind is geostrophic, as the middle equation shows. Fourier transforming the first equations and combining it with the second produces a MSE,

$$i\omega u = ik\phi,$$
$$\frac{\partial \phi}{\partial y} = -\beta y u = -\beta y \frac{k}{\omega}\phi = -\frac{\beta y}{C}\phi,$$

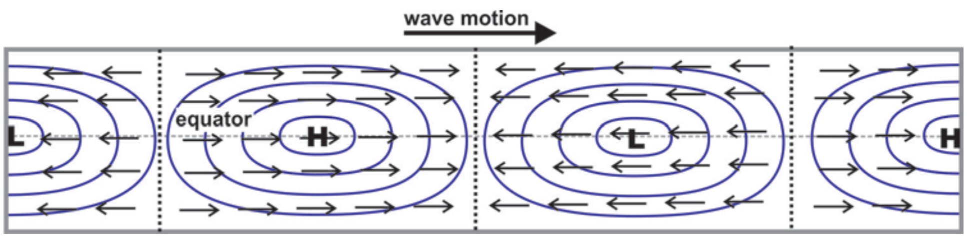
Separating variables and integrating,

$$\frac{d\phi}{\phi} = -\frac{\beta y}{C} dy,$$
$$\ln \phi = -\frac{\beta y^2}{2C} + \text{const.}$$

If at y = 0, ϕ has amplitude A, the complete solution is,

$$\phi(x,y,t) = A\exp\left\{-\frac{\beta y^2}{2C}\right\} e^{i(\omega t - kx)} = A\exp\left\{-\frac{\beta y^2}{2C}\right\} e^{ik(Ct - x)}.$$

Here, we can eliminate the minus root because it would produce solutions that grow without limit away from the equator. Thus these waves have Gaussian meridional structure, have flow strictly in the zonal dierction, and propagate eastward with the shallow-water gravity wave speed. Paradoxically, the zonal flow is in geostrophic balance, even though the waves propagate like gravity waves by divergence ahead (east) of the trough and convergence ahead of the ridge.



In the Madden-Julian Oscillation, baroclinic Kelvin waves are excited by convection in the Indian Ocean. They propagate eastward across the Maritime Continent into the Pacific. It modulates monsoon convection and typhoon formation in the Pacific, and even influences hurricane formation during active seasons in the Atlantic. The MJO was first discovered in rawindsonde data form Christmas Island that showed alternating westerly and easterly flow with a 30-60 day period. Consequently the MJO is sometimes called the 30-60-Day Oscillation.

Kelvin waves also play a role in the oceanic El Niño-Southern Oscillation (ENSO) phenomenon. Normally, the Pacific trade winds push warm waters of the oceanic mixed layer westward against the Maritime Continent. When the trades are interrupted by “westerly wind bursts” warm water surges eastward as a Kelvin wave. It is important to keep in mind that both the MJO- and ENSO- Kelvin-Waves are internal waves so that one needs to replace the external gravity wave speed with an internal one like $C_{1,2} = \pm\sqrt{\sigma g H_1 H_2 / (H_1 + H_2)}$, where σ is the density contrast between layers with depths H_1 and H_2.

The nondimensionalizing factor $(C / \beta)^{1/2}$ defines the width of the ***Equatorial Waveguide*** within which Kelvin waves can propagate. For atmospheric internal waves with a phase speed of 30 m s^{-1}, the equatorial wave guide extends ≈10^{o} north and south of the Equator. For more slowly moving oceanic internal waves with 30 m s−1 phase speed, it extends < 4^{o} from the Equator.