Objective: To present a basic overview of turbulence in the atmosphere.

Reading: Holton, Ch 8, pp 255-275.

Topics:

- Reynolds number and viscosity
- Turbulent flow and Reynolds stresses
- Ekman's solution
- Mixing length and log-wind profiles
- Spectra of turbulence

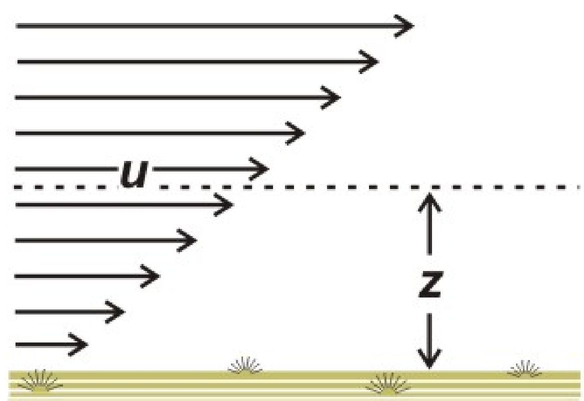
Problems: None

Turbulence is an "…inspiringly complex…superposition of swirls, called eddies, that interact nonlinearly to create quasi-random chaotic motions" that, in turn, interact with a somewhat less chaotic mean flow.

Viscosity and Reynolds Number

We start with (linear) Newtonian viscosity in which the stress, τ, is proportional to the shear of the mean flow,

$$\tau = \upsilon \frac{\partial u}{\partial z}$$



Here ν (Greek nu) is the ***Kinematic Viscosity***. If the density is included, μ = ρν is the ***Dynamic Viscosity***. For molecular viscosity, momentum is carried by Browniain motion of the atoms or molecules of fluid across distances proportional to the mean free path.

The net stress on a thin layer is the difference between that at the top and bottom of the layer in the limit that the layer thickness approaches zero,

$$\frac{\partial \tau}{\partial z} = \frac{\partial}{\partial z}\left(\upsilon \frac{\partial u}{\partial z}\right) = \upsilon \frac{\partial^2 u}{\partial z^2},$$

if ν is constant. Since ∂τ/∂z is proportional to the viscous acceleration, the units of the kinematic viscosity are,

$$\frac{\mathrm{m}}{\mathrm{s}^{-2}} \stackrel{d}{=} \upsilon \frac{\mathrm{m}}{\mathrm{s}} \frac{1}{\mathrm{m}^2} \Rightarrow \upsilon \stackrel{d}{=} \frac{\mathrm{m}^2}{\mathrm{s}}$$

The ratio of inertia "forces," i.e. accelerations, to viscous forces is thus,

$$\mathrm{Re} = \frac{V^2 / L}{\upsilon V / L^2} = \frac{VL}{\upsilon} = \text{Reynolds Number}$$

If Re > 6000, most flows become turbulent. The molecular kinematic viscosity of air is about $1.5 \times 10^{-5}\ \mathrm{m^2\ s^{-1}}$ at 15°C.

In the atmospheric near-surface layer, where V = 10 m s^{-1}, L = 100 m,

$$\mathrm{Re} = \frac{(10\ \mathrm{m\ s^{-1}})(100\ \mathrm{m})}{1.5\times 10^{-5}\ \mathrm{m^2 s^{-1}}} = \frac{1000}{1.5\times 10^{-5}} = 6.66\times 10^7 ,$$

so that the flow is turbulent.

Reynolds Stresses

Horizontal x-momentum and continuity equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

Multiply the top equation by ρ and the bottom by u and add,

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho v u}{\partial y} + \frac{\partial \rho w u}{\partial z} - f\rho v = -\frac{\partial p}{\partial x}$$

If we break the flow into mean and turbulent perturbation flows, $u = \overline{u} + u', v = \overline{v} + v',$ and $w = \overline{w} + w'$, such that (for example), $\overline{uv} = \overline{(\overline{u} + u')(\overline{v} + v')} = \overline{\overline{u}\,\overline{v} + \overline{u}v' + u'\overline{v} + u'v'} = \overline{\overline{u}\,\overline{v}} + \overline{u'v'}$, since, for example, $\overline{u'} = 0$ and $\overline{\overline{u}} = \overline{u}$. Making similar "Reynolds expansions" throughout, neglecting variations of density since the Mach number is small, and expanding the advective terms for the mean flow,

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} + \left[\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{v'u'}}{\partial y} + \frac{\partial \overline{w'u'}}{\partial z}\right] - f\overline{v} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

The terms in square brackets are the Reynolds stresses. One can multiply the Navier-Stokes equations by mean and perturbation velocity components to get prognostic equations for the turbulent energy and Reynolds stresses. This procedure forms the bases of moment-closure turbulence parameterizations. The simplest example is representation of the Reynolds stresses using ***an Eddy Viscosity***, *K*, that is analogous in from with the molecular viscosity, but generally much larger,

$$-\rho\overline{u'w'} = \rho K \frac{\partial \overline{u}}{\partial z}, \text{etc.}$$

K has the same dimensions as ν, but it is generally much larger, ~10^2 m^2 s^{-1}. This formulation is called…

K-Theory.

A higher-order closure might involve obtaining a prognostic (or diagnostic) equation for the turbulent kinetic energy, $E = \frac{1}{2}(u'^2 + v'^2 + w'^2)$ and approximating $K = \ell E^{1/2}$, where ℓ is a "characteristic" length scale.

Ekman Layer

Steady-state, momentum equations in horizontally homogeneous flow with *K*-theory turbulent momentum transport,

$$\frac{1}{\rho}\frac{\partial p}{\partial x} - fv - K\frac{\partial^2 u}{\partial z^2} = 0,$$
$$\frac{1}{\rho}\frac{\partial p}{\partial y} + fu - K\frac{\partial^2 v}{\partial z^2} = 0.$$

Represent, as before, the pressure gradient in terms of the geostrophic wind,

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}, v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}.$$

Substituting,

$$-f(v - v_g) = K\frac{\partial^2 u}{\partial z^2},$$
$$f(u - u_g) = K\frac{\partial^2 v}{\partial z^2}.$$

Adding *i* times the bottom equation to the top equation,

$$-f(v-v_g)+if(u-u_g)=K\frac{\partial^2 u}{\partial z^2}+iK\frac{\partial^2 v}{\partial z^2},$$
$$i[f(u-u_g)+if(v-v_g)]=K\frac{\partial^2 (u+iv)}{\partial z^2},$$
$$if[(u+iv)-(u_g+iv_g)]=K\frac{\partial^2 (u+iv)}{\partial z^2},$$
$$\frac{\partial^2 U}{\partial z^2}-\frac{if}{K}U=-\frac{if}{K}U_g.$$

Where the complex numbers, $U = u + iv$ and $U_g = u_g + iv_g$. This is a classic inhomogeneous ordinary differential equation. It's solution is the superposition the complementary solution of the homogeneous equation and the particular solution, in this case a constant, that also satisfies the inhomogeneous equation. Find the particular solution first.

$$\frac{\partial^2 A}{\partial z^2}-\frac{if}{K}A=0-\frac{if}{K}A=-\frac{if}{K}U_g \quad \Rightarrow A=U_g$$

Assume complementary solutions (to the homogenous equation) of the form $U=Ce^{mz}$, where C is a constant of integration, yet to be determined. Substituting,

$$m^2e^{mz}-\frac{if}{K}e^{mz}=0\Rightarrow m=\pm\sqrt{\frac{if}{K}}=\pm(1+i)\sqrt{\frac{f}{2K}}$$

Since $(1 + i)^2 = 1 + 2i - 1 = 2i$, $\sqrt{i}=2^{-\frac{1}{2}}(1+i)$

Thus, the complete solution is,

$$U(z)=U_g+C\exp\left(\pm\sqrt{\frac{f}{2K}}z\right)\left[\cos\sqrt{\frac{f}{2K}}z\pm i\sin\sqrt{\frac{f}{2K}}z\right],$$

.

We choose the negative root, since we want the solution to be bounded far from the surface. Then we impose a "no slip" surface boundary condition such that $U(0) = 0$. Since the sine is zero and the exponential is one,

$$U(0)=U_g+Ce^{-0}\left[\cos 0-i\sin 0\right]=0.$$

If we choose $C = -U_g$, the solution is zero at the surface and approaches U_g as z becomes large.

$$U(z)=U_g\left\{1-\exp\left(-\sqrt{\frac{f}{2K}}z\right)\left[\cos\left(\sqrt{\frac{f}{2K}}z\right)-i\sin\left(\sqrt{\frac{f}{2K}}z\right)\right]\right\},$$

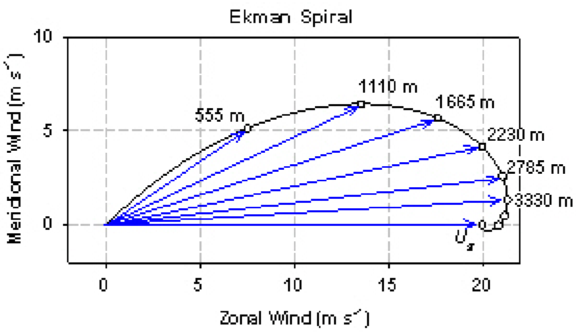
If we set $U_g = u_g + 0i$ to be strictly real so that the geostrophic flow is zonal, the solution becomes,

$$u+iv=u_g\left\{1-\exp\left(-\sqrt{\frac{f}{2K}}z\right)\left[\cos\left(\sqrt{\frac{f}{2K}}z\right)-i\sin\left(\sqrt{\frac{f}{2K}}z\right)\right]\right\},$$

$$u=u_g\left\{1-\exp\left(-\sqrt{\frac{f}{2K}}z\right)\cos\left(\sqrt{\frac{f}{2K}}z\right)\right\}, v=u_g\left\{\exp\left(-\sqrt{\frac{f}{2K}}z\right)\sin\left(\sqrt{\frac{f}{2K}}z\right)\right\}$$

The factor multiplying z is called the inverse Ekman depth, γ, where,

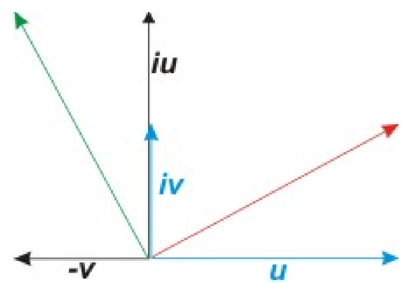
$$\gamma=\frac{1}{\text{Ekman Depth}}=\frac{1}{D_E}=\sqrt{\frac{f}{2K}}\Rightarrow D_E=\sqrt{\frac{2K}{f}}=\sqrt{\frac{2\times10^2}{10^{-4}}}=\sqrt{2\times10^6}=1.4\,\text{km}$$



Near the surface,

$$u=u_g\left[1-(1-\gamma z)\left(1-\frac{\gamma^2z^2}{2}\right)\right]\cong\gamma z,$$
$$v=u_g(1-\gamma z)\gamma z\cong\gamma z$$

So that very near the surface, the wind crosses the isobars at a 45° angle. High above the surface is approaches the geostrophic wind asymptotically. Between these levels the frictionally induced, cross-isobar flow is predominantly toward low pressure. What is the total of this flow integrated over the whole Ekman layer? Note, if $U_g = u_g + iv_g$, $iUg = iu_g - v_g$, which is perpendicular to the original Ug. If we go back to complex notation,



$$U=U_g[1-\exp\{-(1+i)\gamma z\}]$$

The first term in square brackets is the downstream geostrophic flow. Integrating the second term vertically,

$$\int_0^\infty\left(-U_g\exp\{-(1+i)\gamma z\}\right)dz=-\frac{U_g}{(1+i)\gamma}=-\frac{(1-i)U_g}{2\gamma}$$

The downstream geostrophic mass flux is reduced by $Ug/2\gamma$ and there is a mass flux toward low pressure, also equal to $Ug/2\gamma$. The components of the vertically integrated departure from the geostrophic wind are, $\overline{u}=-v_g/2\gamma$ and $\overline{v}=u_g/2\gamma$. From mass continuity, neglecting density variation near the surface,

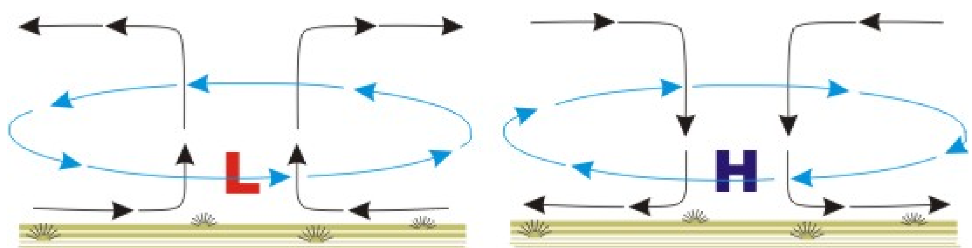
$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right),$$

$$w - w_{sfc} = w = -\int_{sfc}^{0}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dz = -\left(\frac{\partial \overline{u}}{\partial x} - \frac{\partial \overline{v}}{\partial y}\right) = \frac{1}{2\gamma}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) = \frac{\zeta_g}{2\gamma} = \frac{\zeta_g D_E}{2},$$

Where, again, we have assumed that most of the frictional convergence happens at z << *H*. For synoptic-scale weather systems $\zeta_g \sim 10^{-5}$ and $1/\gamma = 1500$ m. Thus $\zeta_g/2\gamma = 0.75$ cm s^{-1}. The rising (or sinking) motion motion has the effect of compressing or stretching vertical vortex tubes of deep circulation systems in the free atmosphere.

$$\frac{\partial \zeta_g}{\partial t} = f\frac{\partial w}{\partial z} \cong -f\frac{w}{H} = -f\frac{\zeta_g}{2\gamma H} = -\frac{f}{2\gamma H}\zeta_g = -\frac{fD_E}{2H}\zeta_g$$

The quantity $2H/fD_E = 2 \times 7000/(10^{-4} \times 1500) \sim 93333$s ~ 1 day is termed the Ekman spin-down time. It is the time for Ekman pumping to reduce large-scale vorticity by a factor of $e^{-1} = 0.3678$.



Mixing Length:

Represent the eddy velocity in the Reynolds as,

$$u' = -\ell\frac{\partial \overline{u}}{\partial z}$$

For isotropic turbulence, *u'* = *w'* so that we can write the Reynolds stress,

$$\tau = -\rho\overline{u'w'} = \rho\ell^2\left(\frac{\partial \overline{u}}{\partial z}\right)^2$$

Very near the surface, the stress is constant because if it changed much with height the accelerations in the thin (low mass) layer would be huge. Since τ has units of density times velocity squared, we define u*, the friction velocity, such that $\rho u^{*2} = \tau$. Thus,

$$\rho u^{*2} = \rho\ell^2\left(\frac{\partial \overline{u}}{\partial z}\right)^2$$

The sizes of the eddies are limited by proximity to the ground. We represent this effect by writing $\ell = kz$, where *k* is von Karman's constant = 0.41. Although derivations like this one seldom mention it, k also allows for imperfect correlation between u' and w'. Substituting and simplifying,

$$u^* = kz\frac{\partial \overline{u}}{\partial z}, \text{ or}$$

$$d\overline{u} = \frac{u^*}{k}\frac{dz}{z}$$

Integrating,

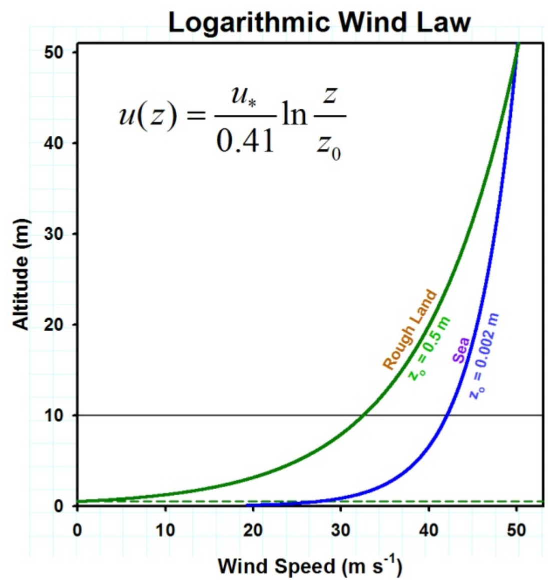
$$\overline{u} = \frac{u^*}{k}\ln z + \text{const}$$

At some level z_0, the mean wind is zero,

$$\overline{u} = \frac{u^*}{k}\ln\frac{z}{z_0}$$

This is the Logarithmic Wind Law. The quantity z_0 is called the roughness length. It characterizes the surface over which the wind blows. It is typically 0.1 to 0.5 the height of the roughness elements themselves. Typical values are:

- Open ocean, z_0 = 2 mm
- Short grass, z_0 = 3-8 mm
- Long grass, z_0 = 6 cm
- Brush, z_0 = 10-50 cm
- Forest or suburbs, z_0 = 1 to 2 m



Alternatively, the surface stress may be represented using the ***Bulk Aerodynamic Formula,***

$$\tau = \rho C_D U_{10}^2, \text{ or } u^{*2} = \frac{\tau}{\rho} = C_D U_{10}^2$$

Combining the Log wind Law and Bulk Aerodynamic Formula and evaluating at 10 m.

$$U_{10} = \frac{\sqrt{C_D}U_{10}}{k}\ln\frac{10\text{m}}{z_0}$$

Canceling the 10-m wind and solving for z_0,

$$z_0 = (10\,\text{m})\exp\left\{-k/\sqrt{C_D}\right\}$$

For example, if C_D = 0.003, z_0 = 0.0056.

Spectra of turbulence:

Nonlinear terms in the 3-D vorticity equation cascade energy to smaller and smaller spatial scales. Ultimately, it ends up with such a small spatial scale that molecular viscosity dissipates it to heat. Dimensional analysis due to *Andrei Nikolaievich Kolmagorov* provides insight into this problem. The dissipation rate per unit mass is ε, which has units of $(m^2\,s^{-2})/s = m^2\,s^{-3}$. We define the spatial scales of turbulence elements in terms of wavenumber, k, with units m^{-1}. We are interested in $E(k)$, the spectrum of energy per unit wavenumber, $(m^2\,s^{-2})/m^{-1} = m^3\,s^{-2}$. A power-law expression for $E(k)$ is,

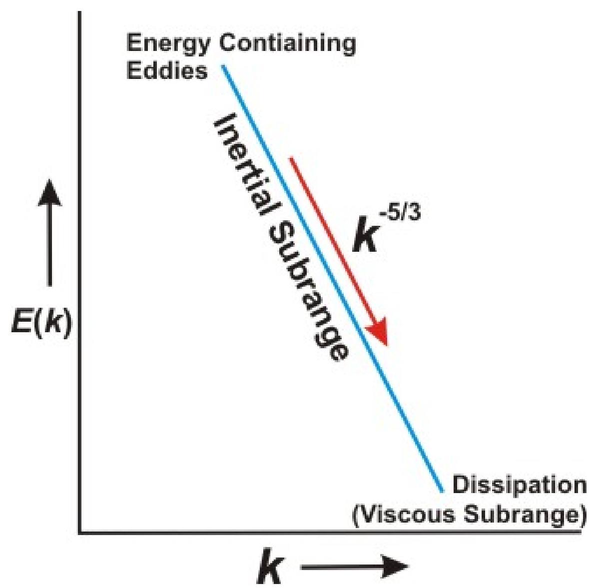
$$E(k) = \alpha_1 \varepsilon^m k^n$$
$$\frac{\mathrm{m}^3}{\mathrm{s}^2} \overset{d}{=} \left(\frac{\mathrm{m}^2}{\mathrm{s}^3}\right)^m \left(\frac{1}{\mathrm{m}}\right)^n$$

The only way to get s^{-2} is if m = 2/3

$$\frac{\mathrm{m}^3}{\mathrm{s}^2} \overset{d}{=} \left(\frac{\mathrm{m}^{4/3}}{\mathrm{s}^2}\right)\left(\frac{1}{\mathrm{m}}\right)^n$$

If n = −5/3, we have $m^{4/3+5/3} = m^3$ on the right, so that,

$$E(k) = \alpha_1 \varepsilon^{2/3} k^{-5/3}$$



For large-scale motions, the flow is nearly two-dimensional. If we multiply ζ times the vorticity equation, we get an Enstrophy, $\overline{\zeta^2}$, conservation equation. But if the turbulent amplitude stays constant as the spatial scale decreases, enstrophy is not constant. The Atmosphere is observed to have two spectral regimes.

In the small-scale regime energy transfers up- and down-scale as described above. There, the spectrum obeys a $k^{-5/3}$ law.

In the large-scale regime, enstrophy cascades both up and down the spectrum. Here we apply Kolmagorov's reasoning with a source of enstophy $\overline{\dot{\zeta}^2} \overset{d}{=} s^{-3}$ and wavenumber $k \overset{d}{=} m^{-1}$ as before we want $E(k) \overset{d}{=} m^3 s^{-2}$, so that

$$E(k) = \alpha_2 \overline{\dot{\zeta}^2}^m k^n$$
$$\frac{\mathrm{m}^3}{\mathrm{s}^2} \overset{d}{=} \left(\frac{1}{\mathrm{s}^3}\right)^m \left(\frac{1}{\mathrm{m}}\right)^n$$

Clearly the only way to satisfy this equation is m = ⅔ and n = −3, such that,

$$E(k) = \alpha_2 \left(\overline{\dot{\zeta}^2}\right)^{2/3} k^{-3}$$

In the large scale, enstrophy-conserving regime spectral energy is inversely proportional to the cube of the wavenumber; in the small-scale regime it is proportional to the −5/3 power of wavenumber. Oddly enough there is observed to be a regime above the energy containing convective regime where energy cascades up-scale along a −5/3 spectum.