Objective: To introduce the linear perturbation method and to apply it to several simple kinds of wave motion.

Reading: CH 5, pp 127-144

Problems: 5.1, 5.2, 5.3, 5.4, and 5.5

Accurate prediction of day-to-day weather requires the full primitive equations, but we can use simplified versions to achieve insight and understanding.

Perturbation analysis changes the fiendishly nonlinear governing equations into linear equations that are easy to solve and describe real phenomena. The process is called "Linearization".

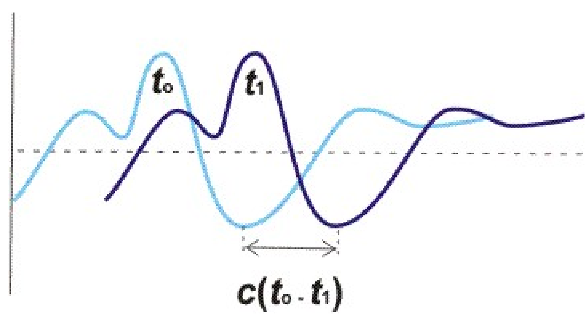
Write, for example, the horizontal velocities as $u=\overline{u}+u'$ and $v=\overline{v}+v'$ and neglect products of the primed (perturbation) quantities with each other but not their products with the overbar (mean) quantities:

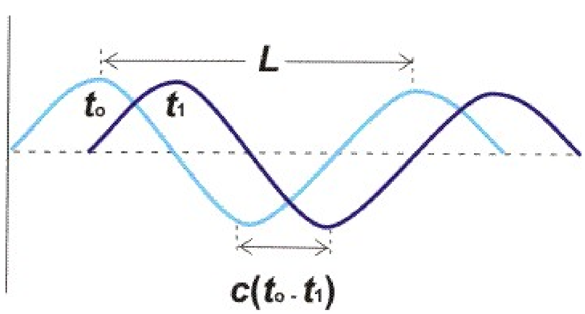
$$u\frac{\partial v}{\partial x}=(\overline{u}+u')\frac{\partial(\overline{v}+v')}{\partial x}=\overline{u}\frac{\partial \overline{v}}{\partial x}+\overline{u}\frac{\partial v'}{\partial x}+u'\frac{\partial \overline{v}}{\partial x}+u'\frac{\partial v'}{\partial x}$$

The $\overline{()}$ quantities are generally described by some sort of balanced flow (geostrophic, thermal-wind) and are often constant in time and space. In this case we write $\overline{u}=-f_0^{-1}\partial\overline{\phi}/\partial y, \overline{v}=0$ and neglect the last term above because it is a small product of prime quantities:

$$u\frac{\partial u}{\partial x}\rightarrow\overline{u}\frac{\partial u'}{\partial x}$$

Most often, we are interested in propagating perturbations. For example, a disturbance that translates without changing shape $\Phi=Af(x-ct)$, where *A* is the disturbance amplitude, *c* is its propagation speed, *x* is a spatial coordinate and t is time.





A special case is a sinusoidal wave with length *L* and period *T*:

$$\Phi=A\cos(kx-\omega t)+B\sin(kx-\omega t),$$

where $k = 2\pi/L$ and $\omega = 2\pi/T$ are the wavenumber and frequency. The phase speed of the wave is ω/k or, equivalently L/T. A reason for being interested in sine waves is that any arbitrary periodic function, $f(x) = f((x + L)$ or $f(t) = f((t + T)$, can be represented as a Fourier series—an infinite sum of sines and cosines:

$$f(x) = A_0 + \sum_1^{\infty} A_n \cos\frac{2n\pi x}{L} + B_n \sin\frac{2n\pi x}{L}$$

Here A_n and B_n are the Fourier coefficients. We find them by multiplying the function to be represented by sin $2\pi nx/L$ or cos $2\pi nx/L$ and integrating the product over a complete wavelength. Since

$$\int_{-L/2}^{L/2} \sin\frac{2n\pi x}{L}\sin\frac{-2m\pi x}{L}dx = \begin{cases} 0, \text{when } n \neq m \\ L, \text{when } n = m \end{cases}.$$

This procedure isolates the part of the function that "projects onto" each Fourier component. The same relation works for cosines, but not for mixtures of sines and cosines, where the integral is always zero. Thus, the formulas for the Fourier coefficients are:

$$A_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\cos\frac{2n\pi x}{L}dx, \quad B_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\sin\frac{2n\pi x}{L}dx,$$

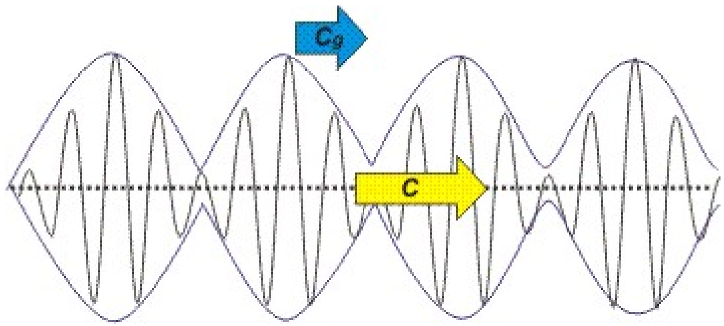
Such that $f_n = A_n \cos\frac{2n\pi x}{L} + B_n \sin\frac{2n\pi x}{L}$ is the nth Fourier component, or the nth harmonic.

One can represent sines and cosines using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, where $i = \sqrt{-1}$ or $i^2 = -1$. We can thus write the nth Fourier component as $f_n(x) = \text{Re}\{C_n \exp(-2n\pi ix/L)\} =$ $\text{Re}\{A_n \cos 2n\pi x/L + B_n \sin 2n\pi x/L + i[B_n \cos 2n\pi x/L - A_n \sin 2n\pi x/L]\}$, where $C_n = A_n + iB_n$ is the complex amplitude, or equivalently $A_n = \text{Re}\{C_n\}$ and $B_n = \text{Im}\{C_n\}$. Note that $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$.

Dispersion and Group Velocity:

Wave motion is said to be **Dispersive** when frequency is a function of wavenumber (other than directly proportional to k). In other words, waves are dispersive when the propagation speed, c, is a function of wavenumber. A dispersive wavetrain will change shape as it propagates because the Fourier components move with different speeds. Consider a "superposition" of two wavetrains with slightly different wavenumbers and frequencies.

$$\begin{aligned}
\Psi(x,t) &= \exp i[(k+\delta k)x - (\omega + \delta\omega)t] + \exp i[(k-\delta k)x - (\omega - \delta\omega)t] \\
&= \exp i(kx - \omega t)[\exp i(\delta kx - \delta\omega t) + \exp - i(\delta kx - \delta\omega t)] \\
&= 2\exp ik[x - (\omega/k)t]\{\exp i\delta k[x - (\delta\omega/\delta k)t] + \exp - i\delta k[x - (\delta\omega/\delta k)t]\}/2 \\
&= 2\exp ik[x - (\omega/k)t]\cos\delta k[x - (\delta\omega/\delta k)t] \\
&= 2\exp ik[x - (\omega/k)t]\cos\delta k[x - (\partial\omega/\partial k)t] = 2\exp ik[x - ct]\cos\delta k[x - c_g t]
\end{aligned}$$



Here $\delta k << k$ and $\delta\omega << \omega$. The cosine function represents the modulation or envelope of the wave "packets". It moves with the group velocity $c_g = \partial\omega/\partial k$. The imaginary exponential represents the carrier. It moves with the phase velocity $c = \omega/k$. Energy moves with wave packets and generally has a different speed (or even direction) of propagation from the carrier waves. A wave phenomenon where c = constant is said to be "nondispersive". In that case, energy and phase move with the same speed.

An example of wave motion, one-dimensional **Sound, or Acoustic, Waves**. Start with a slightly different form of the primitive governing equations. We use the x momentum equation with no rotation, mass continuity, the original form of the first law of thermodynamics set in terms of the time derivatives of temperature and specific volume, and the equation of state for dry air:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = -\rho\frac{\partial u}{\partial x},$$
$$c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt} = 0,$$
$$RT = p/\rho = p\alpha$$

We want to use the thermodynamic relations to eliminate temperature in the 1st law and to replace density in the continuity equation with pressure. Remember that $\alpha = 1/\rho$ so that $D\alpha/Dt = -1/\rho^2\, D\rho/Dt$. From the gas law,

$$R\frac{DT}{Dt} = \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} \Rightarrow \frac{DT}{Dt} = \frac{1}{R}\left(\frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}\right)$$

Substituting into the 1st Law,

$$\frac{c_v}{R}\left(\frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}\right) - \frac{p}{\rho^2}\frac{D\rho}{Dt} = 0$$

Rearranging:

$$\frac{c_v}{R}\frac{1}{\rho}\frac{Dp}{Dt} = \frac{c_v}{R}\frac{p}{\rho^2}\frac{D\rho}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{c_v + R}{R}\frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{c_p}{R}\frac{p}{\rho^2}\frac{D\rho}{Dt}$$

Which may be rewritten as:

$$\frac{1}{p}\frac{Dp}{Dt}=\frac{c_v}{c_p}\frac{1}{\rho}\frac{D\rho}{Dt}=\gamma\frac{1}{\rho}\frac{D\rho}{Dt},$$

where $\gamma \equiv c_p/c_v$. From continuity:

$$\frac{1}{\rho}\frac{D\rho}{Dt}=\frac{1}{\gamma}\frac{1}{p}\frac{Dp}{Dt}=-\frac{\partial u}{\partial x}$$

Make a perturbation analysis by writing $u=\overline{u}+u'$, $p=\overline{p}+p'$, $\rho=\overline{\rho}+\rho'$, where the mean quantities are constant in time and space. The momentum and continuity equations become:

$$\frac{\partial(\overline{u}+u')}{\partial t}+(\overline{u}+u')\frac{\partial(\overline{u}+u')}{\partial x}+\frac{1}{\overline{\rho}+\rho'}\frac{\partial(\overline{p}+p')}{\partial x}=0$$
$$\frac{\partial(\overline{p}+p')}{\partial t}+(\overline{u}+u')\frac{\partial(\overline{p}+p')}{\partial x}+\gamma(\overline{p}+p')\frac{\partial(\overline{u}+u')}{\partial x}=0$$

We can write $1/(\overline{\rho}+\rho')=1/\overline{\rho}(1+\rho'/\overline{\rho})\approx(1/\overline{\rho})(1-\rho'/\overline{\rho}+...)$. Since mean quantities are constant, we have:

$$\frac{\partial u'}{\partial t}+(\overline{u}+u')\frac{\partial u'}{\partial x}+\frac{1}{\overline{\rho}}\left(1-\frac{\rho'}{\rho}\right)\frac{\partial p'}{\partial x}=0$$
$$\frac{\partial p'}{\partial t}+(\overline{u}+u')\frac{\partial p'}{\partial x}+\gamma(\overline{p}+p')\frac{\partial u'}{\partial x}=0$$

Finally, neglecting the products of prime quantities gives us:

$$\frac{\partial u'}{\partial t}+\overline{u}\frac{\partial u'}{\partial x}+\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x}=0$$
$$\frac{\partial p'}{\partial t}+\overline{u}\frac{\partial p'}{\partial x}+\gamma\overline{p}\frac{\partial u'}{\partial x}=0$$

When we take the linearized Lagrangian derivative of the momentum equation, we get:

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 u'=-\frac{1}{\overline{\rho}}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial p'}{\partial x}=-\frac{1}{\overline{\rho}}\frac{\partial}{\partial x}\left(\frac{\partial p'}{\partial t}+\overline{u}\frac{\partial p'}{\partial x}\right)=$$
$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial x}\left(\gamma\overline{p}\frac{\partial u'}{\partial x}\right)=\frac{\gamma\overline{p}}{\overline{\rho}}\frac{\partial^2 u'}{\partial x^2}=\gamma R\overline{T}\frac{\partial^2 u'}{\partial x^2}$$

We now assume that the solution is a single Fourier component written in exponential notation:

$$u=A\exp\{-i(\omega t-kx)\}$$

And substitute into the combined equation:

$$A[i(\omega - k\overline{u})]^2 \exp\{-i(\omega t - kx)\} = \gamma R\overline{T}(ik)^2 A\exp\{-i(\omega t - kx)\}$$

Which readily solves to:

$$\omega = k(\overline{u} \pm \sqrt{\gamma R\overline{T}})$$

The quantity $(\gamma RT)^{1/2}$ is the speed of sound, or acoustic wave speed. Recall that $\gamma = c_p/c_v$ = 1004 J K^{-1} kg^{-1}/717 J K^{-1} kg^{-1} = 1.400 and R = 287 J K^{-1} kg^{-1}. Dimensionally a J K^{-1} kg^{-1} = Kg m^2/s^2 $K^{-1}kg^{-1}$ = m^2/s^2 K^{-1}. At the standard-atmosphere surface temperature 288K:

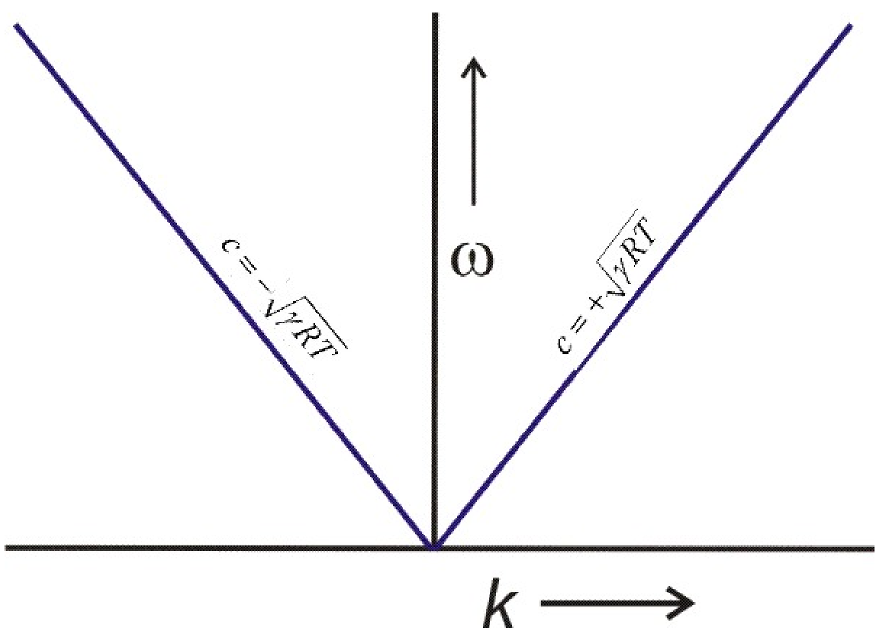
$c_A = (1.400 \times 287 \times 288)^{1/2} = 340$ m s^{-1}

At the standard-atmosphere tropopause temperature 216.5K:

$c_A = (1.400 \times 287 \times 216.5)^{1/2} = 295$ m s^{-1}

So that the speed of sound decreases with height in a normally lapsing atmosphere, but increases in an inversion.

If we plot a Dispersion Diagram, showing ω as a function of *k*, we see that the frequency increases linearly k, which is equivalent to constant phase speed.



Are these waves dispersive? What is the group velocity?