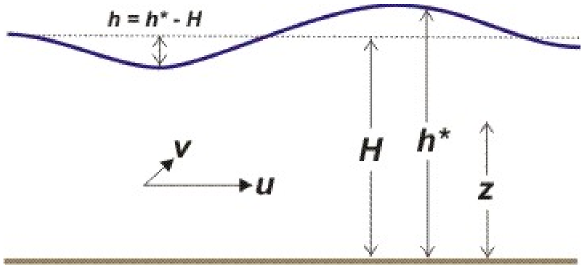
Objective: To apply the perturbation method to meteorologically significant problems.

Reading: Continue from LEC 1A

Problems: Continue from LEC 1A

Shallow-Water Gravity Waves:

Consider a one-dimensional shallow-water system with no background rotation:



$$\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x} = -g \frac{\partial h^*}{\partial x},$$
$$\frac{\partial h^*}{\partial t} + u^* \frac{\partial h^*}{\partial x} = -h^* \frac{\partial u}{\partial x}$$

Make a perturbation expansion of the system with $u^* = U + u$, $h^* = H + h$, where U and H are spatially and temporally constant, and $h << H$ and $u << U$.

$$\frac{\partial (U+u)}{\partial t} + (U+u) \frac{\partial (U+u)}{\partial x} = -g \frac{\partial (H+h)}{\partial x},$$
$$\frac{\partial (H+h)}{\partial t} + (U+u) \frac{\partial (H+h)}{\partial x} = -(H+h) \frac{\partial (U+u)}{\partial x}$$

Dropping the derivatives of the constant mean quantities and expanding the products:

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x},$$
$$\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + u \frac{\partial h}{\partial x} = -H \frac{\partial u}{\partial x} - h \frac{\partial u}{\partial x}$$

The last terms before the equal sign in both equations, and the final term on the right in the contiumuty equation are negligible because they are products of perturbation quantities. The resulting linear governing equations are:

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x},$$
$$\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} = -H \frac{\partial u}{\partial x}$$

Take $\partial / \partial t + U \partial / \partial x$ of the continuity equation (works just as well if we apply the operator to the momentum equation):

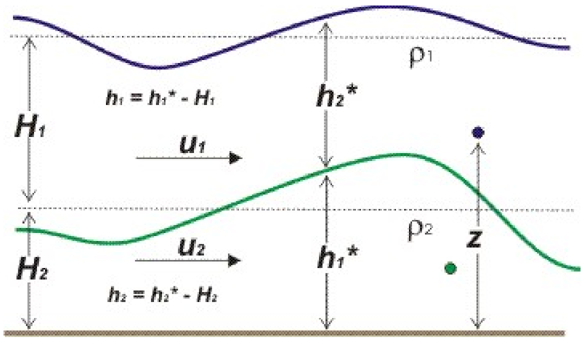
$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2 h=H\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u=gH\frac{\partial^2 h}{\partial x^2}$$

We now assume an exponential solution of the form $h(x, t) = \exp\{-i(\omega t - kx)\}$. With this substitution, the equation becomes:

$$[i(\omega-kU)]^2=gH(ik)^2\ ,$$

or

$$\omega=k\left(U\pm\sqrt{gH}\right)$$



Suppose that the fluid has two layers with different spatially and temporally constant densities ρ_1 and ρ_2, but the same spatially and temporally constant mean flow, U, in both. The pressure in the upper layer at height z_1 above the bottom is $g\ \rho_1(h_1^* + h_2^* - z_1)$; that in the lower layer at height z_2 above the bottom is $g\ [\rho_1 h_1^* + \rho_2 (h_2^* - z_2)]$. The nonlinear governing equations are:

$$\begin{aligned}
\rho_1\left[\frac{\partial}{\partial t}+(U+u_1)\frac{\partial}{\partial x}\right](U+u_1)&=-g\frac{\partial}{\partial x}\left[\rho_1(H_1+h_1+H_2+h_2-z_1)\right]\\
\left[\frac{\partial}{\partial t}+(U+u_1)\frac{\partial}{\partial x}\right](H_1+h_1)&=-(H_1+h_1)\frac{\partial(U+u_1)}{\partial x}\\
\rho_2\left[\frac{\partial}{\partial t}+(U+u_2)\frac{\partial}{\partial x}\right](U+u_2)&=-g\frac{\partial}{\partial x}\left[\rho_1(H_1+h_1)+\rho_2(H_2+h_2-z_2)\right]\\
\left[\frac{\partial}{\partial t}+(U+u_2)\frac{\partial}{\partial x}\right](H_2+h_2)&=-(H_2+h_2)\frac{\partial(U+u_2)}{\partial x}
\end{aligned}$$

We linearize by dropping derivatives of U, H_1, H_2, z_1, and z_2, which are zero and by dropping products of lower-case subscripted quantities which are small.

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u_1&=-g\frac{\partial(h_1+h_2)}{\partial x}\\
\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)h_1&=-H_1\frac{\partial u_1}{\partial x}\\
\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u_2&=-g\frac{\partial}{\partial x}\left(\frac{\rho_1}{\rho_2}h_1+h_2\right)\\
\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)h_2&=-H_2\frac{\partial u_2}{\partial x}
\end{aligned}$$

The factor in parentheses on right-hand side of the momentum equation for the lower layer may be rewritten $[h_1(\rho_1/\rho_2)+h_2]=[h_1+h_2-h_1(\rho_2-\rho_1)/\rho_2+]=[h_1+h_2-\sigma h_1]$, where $\sigma=(\rho_2-\rho_1)/\rho_2$. With this substitution, the lower-level momentum equation becomes:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u_2=-g\frac{\partial}{\partial x}\left(h_1+h_2-\sigma h_1\right)$$

Adding the two mass-continuity equations produces:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)(h_1+h_2)=-H_1\frac{\partial u_1}{\partial x}-H_2\frac{\partial u_2}{\partial x}$$

Taking the linearized Lagrangian derivative of the summed continuity equations:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2(h_1+h_2)&=-H_1\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u_1-H_2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u_2\\&=-H_1\frac{\partial^2}{\partial x^2}[-g(h_1+h_2)]-H_2\frac{\partial^2}{\partial x^2}[-g(h_1+h_2-\sigma h_1)]\\&=\ g(H_1+H_2)\frac{\partial^2}{\partial x^2}(h_1+h_2)-g\sigma H_2\frac{\partial^2}{\partial x^2}h_1\end{aligned}$$

Taking the linearized Lagrangian derivative of the upper-layer continuity equation:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2h_1=-H_1\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u_1=-H_1\frac{\partial}{\partial x}\left[-g\frac{\partial}{\partial x}(h_1+h_2)\right]=gH_1\frac{\partial^2(h_1+h_2)}{\partial x^2}$$

Differentiating the combined continuity equations again:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^4(h_1+h_2)&=g(H_1+H_2)\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\frac{\partial^2(h_1+h_2)}{\partial x^2}-g\sigma H_2\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2h_1\\&=g(H_1+H_2)\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\frac{\partial^2(h_1+h_2)}{\partial x^2}-g\sigma H_2\frac{\partial^4}{\partial x^4}[gH_1(h_1+h_2)]\\&=g(H_1+H_2)\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\frac{\partial^2(h_1+h_2)}{\partial x^2}-g^2\sigma H_1H_2\frac{\partial^4(h_1+h_2)}{\partial x^4}\end{aligned}$$

or

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^4(h_1+h_2)-g(H_1+H_2)\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\frac{\partial^2(h_1+h_2)}{\partial x^2}+g^2\sigma H_1H_2\frac{\partial^4(h_1+h_2)}{\partial x^4}=0$$

As before, assume imaginary exponential solutions, $h_1+h_2=A\exp\{-i(\omega t-kx)\}$. Upon substitution, we cancel the exponential and amplitude leaving an expression that is quadratic in both $[-i(\omega-kU)]^2$ and $(ik)^2$. Note that the former expression is the square of the Doppler-shifted frequency.

$$[-i(\omega-kU)]^4-g(H_1+H_2)[-i(\omega-kU)]^2(ik)^2+g^2\sigma H_1H_2(ik)^4=0$$

Squaring out all of the *i*s, yields:

$$(\omega-kU)^4-g(H_1+H_2)(\omega-kU)^2k^2+g^2\sigma H_1H_2k^4=0$$

Which is quadratic in the square of the Doppler-shifted frequency and may be solved using the conventional algebraic formula:

$$\begin{aligned}(\omega-kU)^2&=\tfrac{1}{2}\left[g(H_1+H_2)k^2\pm\sqrt{g^2(H_1+H_2)^2k^4-4g^2\sigma H_1H_2k^4}\right]\\&=\tfrac{1}{2}g(H_1+H_2)\left[1\pm\sqrt{1-4\frac{\sigma H_1H_2}{(H_1+H_2)^2}}\right]\end{aligned}$$

Since σ << 1, we may expand the radical in a Taylor series,

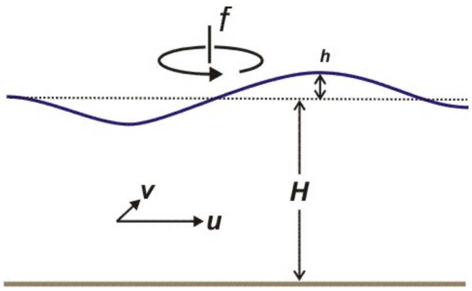
$$(\omega-kU)^2=\tfrac{1}{2}g(H_1+H_2)\left[1\pm\left(1-2\frac{\sigma H_1H_2}{(H_1+H_2)^2}\right)\right]\text{, or}$$

$$\omega=k\left(U\pm\sqrt{g(H_1+H_2)}\right),\quad k\left(U\pm\sqrt{\frac{\sigma gH_1H_2}{(H_1+H_2)^2}}\right)$$

The first solution represents external gravity waves propagating on the free surface as though the fluid had constant density and depth equal to the combined thicknesses of the two layers. The second solution represents internal gravity wave propagating on the internal density boundary as though the surface was a rigid lid. The ratios of the thicknesses appear here because convergence in one layer must be compensated by divergence in the other. The quantity σ*g* is sometimes called the reduced gravity because the relatively small density contrast reduces the gravitational restoring force. Internal waves always propagate much more slowly than external ones. Examples of internal gravity waves include waves on the thermocline in the sea and on inversions in the atmosphere. Solutions for continuously varying stable density stratifications also exist. Another way to solve the quadratic equation for the squared Doppler-shifted frequency is to recognize that, for high-frequency waves, the last term is small compared with the first two terms, and, for low-frequency waves, the first term is negligible with respect to the middle and last terms.

Gravity Waves With Rotation

Here we consider two-dimensional shallow-water waves on an *f* plane with mean flow in the zonal direction only, using linearized equations as before:



$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)u-fv=-g\frac{\partial h}{\partial x},$$
$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)v+fu=-g\frac{\partial h}{\partial y},$$
$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)h=-H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right).$$

Taking the linearized Lagrangian derivative of the zonal momentum equation:

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 u-f\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)v=-g\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial h}{\partial x}$$
$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 u-f\left(-fu-g\frac{\partial h}{\partial y}\right)=-g\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial h}{\partial x}$$
$$\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+f^2\right]u=-g\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial h}{\partial x}+f\frac{\partial h}{\partial y}\right]$$

Similarly for the meridional momentum equation:

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 v+f\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)u=-g\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial h}{\partial y}$$
$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 u+f\left(fv-g\frac{\partial h}{\partial x}\right)=-g\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial h}{\partial y}$$
$$\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+f^2\right]v=-g\left[-f\frac{\partial h}{\partial x}+\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial h}{\partial y}\right]$$

Now, differentiate the continuity equation twice and add f^2 to it:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+f^2\right]h&=-H\left\{\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+f^2\right]u+\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+f^2\right]v\right\}\\
&=-H\left\{-g\frac{\partial}{\partial x}\left[f\frac{\partial h}{\partial y}+\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial h}{\partial x}\right]-g\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial h}{\partial y}-f\frac{\partial h}{\partial x}\right]\right\}\\
&=-gH\left\{-f\frac{\partial^2 h}{\partial x\partial y}-\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial^2 h}{\partial x^2}-\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial^2 h}{\partial y^2}+f\frac{\partial h}{\partial x\partial y}\right\}\\
&=gH\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)\end{aligned}$$

We can drop the outside Lagrangian derivative either because the quantities inside are conserved and we may scale their value to be zero or because we recognize that after Fourier transformation it will become a multiplicative factor that can be cancelled.

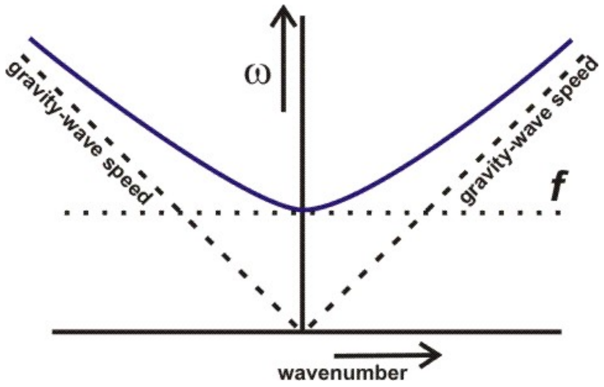
$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 h-gH\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)+f^2 h=0$$

Fourier transforming $h=A\exp\{-i(\omega t-kx-\ell y)\}$:

$$-(\omega+k\overline{u})^2+gH(k^2+\ell^2)+f^2=0$$

So that:

$$\omega=k\overline{u}\pm\sqrt{gH(k^2+\ell^2)+f^2}$$



These are dispersive waves in which *f* defines the lowest possible frequency. Phase velocities are:

$$c_x=\frac{\omega}{k}=\overline{u}\pm\frac{1}{k}\sqrt{gH\left(k^2+\ell^2\right)+f^2},$$
$$c_y=\frac{\omega}{\ell}=\frac{k}{\ell}\overline{u}\pm\frac{1}{\ell}\sqrt{gH\left(k^2+\ell^2\right)+f^2}$$

Since the wavenumbers are inversely proportional to the wavelengths, the waves will appear to move fastest in the direction where they are longest. That is if $k \gg \ell$, $c_y \gg c_x \simeq (gH)^{1/2}$ for $\omega \ll f$. The group velocities are:

$$c_{gx}=\frac{\partial\omega}{\partial k}=\overline{u}\pm\frac{gHk}{\sqrt{gH(k^2+\ell^2)+f^2}},$$
$$c_{gy}=\frac{\partial\omega}{\partial\ell}=\pm\frac{gH\ell}{\sqrt{gH(k^2+\ell^2)+f^2}},$$

Note that for long wave (small *k* and *ℓ*) the frequency approaches *f* and the group velocities (in still air) approaches gHk/f and gHℓ/f. For short waves (large *k* and *ℓ*) the group and phase speeds approach the phase velocity for one-dimensional shallow-water waves. The direction of the group velocitiy is toward the larger wavenumber, in contrast with the phase velocity directions.

Another Look at Inertia Oscillations:

Rewrite the momentum equations with a non-uniform, geostrophic zonal wind:

$$\frac{Du}{Dt}=f_0 v=f_0\frac{Dy}{Dt},$$
$$\frac{Dv}{Dt}=-f_0(u-u_q)$$

When a parcel of air initially moving with the QG flow is displaced meridionally from an initial latitude y_0 by a distance δy across the QG zonal flow:

$$u(y_0+\delta y)=u_g(y_0)+f_0\int\frac{Dy}{Dt}dt=u_g(y_0)+f_0\delta y$$

The meridional variation of the QG flow can be represented using a Taylor series.

$$u_g(y_0+\delta y)\simeq u_g(y_0)+\frac{\partial u_g}{\partial y}\delta y+\ldots$$

Combining these expressions with the meridional momentum equation:

$$\begin{aligned}\frac{Dv}{Dt}=\frac{D^2(\delta y)}{Dt^2}&=-f_0\left[u_g(y_0)+f_0\delta y-\left(u_g(y_0)+\frac{\partial u_g}{\partial y}\right)\right]\\&=-f_0\left[f_0-\frac{\partial u_g}{\partial y}\right]\delta y\end{aligned}$$

What we have here is a Lagrangian expression for the displacement of the parcel. We can assume imaginary exponential solutions for $\delta y = Ae^{-i\omega t}$.

$$-\omega^2=-f_0\left(f_0-\frac{\partial u_g}{\partial y}\right)$$

When $\partial u_g/\partial y = 0$, $\omega = \underline{+} f_0$, but as we saw earlier, only anticyclonic inertial oscillations can happen in reality. If $\partial u_g/\partial y > 0$, but $\partial u_g/\partial y < f_0$, or if $\partial u_g/\partial y < 0$, there will also be inertia oscillation with frequencies lower than f in the former case and higher than f in the latter. When $\partial u_g/\partial y > f_0$, the frequency becomes $\omega=\pm i\sqrt{\partial u_g/\partial y-f_0)}$ and the displacements described by the positive root grow exponentially with time. This situation is said to be **"Inertially Unstable"**.