Internal Gravity Waves in a Stratified Fluid

Governing equations for two-dimensional waves in the *x*–*z* plane:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g,$$
$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+w\frac{\partial \theta}{\partial z}=0,$$
$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0.$$

These motions can be non-hydrostatic, but consider a hydrostatic basic mass distribution:

$$\frac{1}{p}\frac{\partial p}{\partial z}=-\frac{g}{RT},$$
$$\frac{1}{p}\left(\frac{p}{p_0}\right)^{R/c_p}\frac{\partial p}{\partial z}=-\frac{g}{RT}\left(\frac{p}{p_0}\right)^{R/c_p}$$

Here we have multiplied both sides by $(p/p_0)^{R/cp}$. Rearranging and multiplying both sides by R/c_p:

$$\frac{R}{c_p}\frac{1}{p}\left(\frac{p}{p_0}\right)^{R/c_p}\frac{\partial p}{\partial z}=\frac{\partial}{\partial z}\left(\frac{p}{p_0}\right)^{R/c_p}=\frac{\partial \pi}{\partial z}=-\frac{g}{c_pT\left(\frac{p_0}{p}\right)^{R/c_p}}=-\frac{g}{c_p\theta}$$

Here $(p/p_0)^{R/c_p}$ is the "Exner function". It is an algebraically convenient replacement for pressure in problems such as this one, just as potential temperature, θ, is an equally convenient replacement for temperature. Let's look at the right side of the *x*-momentum equation:

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{RT}{p}\frac{\partial p}{\partial x}=c_pT\left(\frac{p_0}{p}\right)^{R/c_p}\frac{R}{c_p}\frac{1}{p}\left(\frac{p}{p_0}\right)^{R/c_p}\frac{\partial p}{\partial x}=c_p\theta\frac{\partial \pi}{\partial x}.$$

Similarly in the vertical:

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=c_p\theta\frac{\partial \pi}{\partial z}$$

So that the momentum equations become:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-c_p\theta\frac{\partial \pi}{\partial x},$$
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-c_p\theta\frac{\partial \pi}{\partial z}-g,$$

We now do a perturbation expansion where $u=\overline{u}+u'(x,z,t), w=w'(x,z,t), \pi=\overline{\pi}(z)+\pi'(x,z,t),$ and $\theta=\overline{\theta}(z)+\theta'(x,z,t).$ The mean-state potential temperature are in hydrostatic balance.

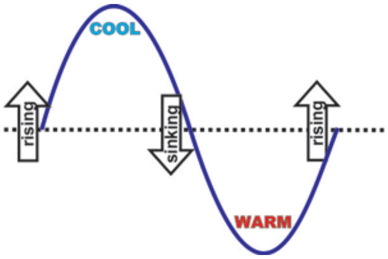
$$
\begin{aligned}
&\frac{\partial u'}{\partial t}+\overline{u}\frac{\partial u'}{\partial x}=-c_p\overline{\theta}\frac{\partial \pi'}{\partial x},\\
&\frac{\partial w'}{\partial t}+\overline{u}\frac{\partial w}{\partial x}=-c_p\overline{\theta}\frac{\partial \overline{\pi}}{\partial z}-c_p\theta'\frac{\partial \overline{\pi}}{\partial z}-c_p\overline{\theta}\frac{\partial \pi'}{\partial z}-g,\\
&\frac{\partial \theta'}{\partial t}+\overline{u}\frac{\partial \theta'}{\partial x}+w'\frac{\partial \overline{\theta}}{\partial z}=0,\\
&\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0.
\end{aligned}
$$

Here, I have deliberately left the product of the mean term on the right side of the vertical momentum equation because it requires attention. Substituting from the mean hydrostatic relation:

$$
\begin{aligned}
-c_p\overline{\theta}\frac{\partial \overline{\pi}}{\partial z}-c_p\theta'\frac{\partial \overline{\pi}}{\partial z}-c_p\overline{\theta}\frac{\partial \pi'}{\partial z}-g &= -c_p\overline{\theta}\left(-\frac{g}{c_p\overline{\theta}}\right)-c_p\theta'\left(-\frac{g}{c_p\overline{\theta}}\right)-c_p\overline{\theta}\frac{\partial \pi'}{\partial z}-c_p\theta'\frac{\partial \pi'}{\partial z}-g\\
&=+g-\theta'\left(-\frac{g}{\overline{\theta}}\right)-c_p\overline{\theta}\frac{\partial \pi'}{\partial z}-g=g\frac{\theta'}{\overline{\theta}}-c_p\overline{\theta}\frac{\partial \pi'}{\partial z}=b-c_p\overline{\theta}\frac{\partial \pi'}{\partial z}
\end{aligned}
$$

Where $b\equiv g(\theta'/\overline{\theta})$ is the ***Buoyancy***. We can rewrite the thermodynamic energy equation in terms of buoyancy by multiplying by g and dividing through by $\overline{\theta}$:

$$
\frac{\partial}{\partial t}\left(g\frac{\theta'}{\overline{\theta}}\right)+\overline{u}\frac{\partial}{\partial x}\left(g\frac{\theta'}{\overline{\theta}}\right)+w'\frac{g}{\overline{\theta}}\frac{\partial \overline{\theta}}{\partial z}=\frac{\partial b}{\partial t}+\overline{u}\frac{\partial b}{\partial x}+w'N^2=0,
$$



where $N^2=(g/\overline{\theta})(\partial\overline{\theta}/\partial z)$ is the Brunt Väisälä, or buoyancy, frequency. The Brunt Väisälä oscillation is vertical with a period of about ten minutes. Initially upward moving air with the same temperature as its surroundings becomes colder than the air around it through adiabatic expansion. As a result it accelerates downward, reversing its upward motion so that is starts sinking back toward its starting point. By the time it reaches its initial level it has downward momentum so that it passes below its starting point and warms through adiabatic compression. As a result its descending motion decelerates, ultimately leading to a return to the initial level with enough upward momentum to carry the air upward into another cycle. In the standard atmosphere, the Brunt Väisälä oscillation has a period of about 10 minutes. If we omit the primes for notational simplicity, the final form of the governing equations is:

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+\overline{u}\frac{\partial u}{\partial x}=-c_p\overline{\theta}\frac{\partial \pi}{\partial x},\\
&\frac{\partial w}{\partial t}+\overline{u}\frac{\partial w}{\partial x}=b-c_p\overline{\theta}\frac{\partial \pi}{\partial z},\\
&\frac{\partial b}{\partial t}+\overline{u}\frac{\partial b}{\partial x}+N^2w=0,\\
&\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0.
\end{aligned}
$$

Taking the linearized Lagrangian derivative of the vertical momentum equation and substituting from the buoyancy equation,

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 w &=\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)b-c_p\overline{\theta}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial \pi}{\partial z}\\
\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 w &=-N^2w-c_p\overline{\theta}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial \pi}{\partial z}
\end{aligned}
$$

Or,

$$
\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]w=-c_p\overline{\theta}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial \pi}{\partial z}
$$

This expression is like the “Magic Operators” that we obtained for inertial oscillation in rotational shallow-water internal gravity waves, but for vertical buoyancy oscillations. Applying the new Magic operator to the continuity equation:

$$
\begin{aligned}
&\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)=\\
&\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial u}{\partial x}+\frac{\partial}{\partial z}\left[-c_p\overline{\theta}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial \pi}{\partial z}\right]=0.
\end{aligned}
$$

Here we make the Boussinesq Approximation, in which variations in density (θ is a proxy here) are neglected except where multiplied by gravity. The idea is that variations of density contributing to inertia are small, but not those leading to buoyancy. Thus we can rewrite the above equation as:

$$
\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial u}{\partial x}-c_p\overline{\theta}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial^2 \pi}{\partial z^2}=0.
$$

We then take the individual derivative again and substitute from the horizontal momentum equation.

$$\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial u}{\partial x}-c_p\bar{\theta}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial^2\pi}{\partial z^2}=$$

$$\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u-c_p\bar{\theta}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial^2\pi}{\partial z^2}=$$

$$\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\left(-c_p\bar{\theta}\frac{\partial^2\pi}{\partial x^2}\right)-c_p\bar{\theta}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\frac{\partial^2\pi}{\partial z^2}=0.$$

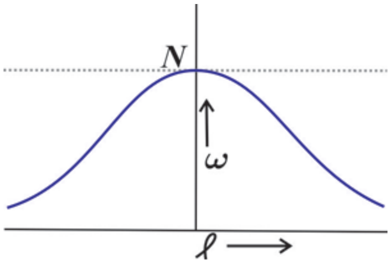
We cancel $-c_p\bar{\theta}$ and, as before, seek solutions of the form $\pi = A\exp i(\omega t-kx-\ell z)$:

$$\left[-\left(\omega-k\bar{u}\right)^2+N^2\right]\left(k^2\right)-\left(\omega-k\bar{u}\right)^2\ell^2=0,$$

which may be solved for the frequency to yield the dispersion relation:

$$\omega = k\bar{u}\pm\sqrt{\frac{N^2k^2}{k^2+\ell^2}}.$$

What is the wave frequency when its horizontal wavelength is much less than its vertical wavelength ($k >> \ell$)? What is the frequency when the horizontal wavelength is much greater than the vertical wavelength ($k << \ell$)?



The horizontal and vertical phase velocities are:

$$c_x=\frac{\omega}{k}=\bar{u}\pm\sqrt{\frac{N^2}{k^2+\ell^2}}=\bar{u}\pm\frac{N}{\sqrt{k^2+\ell^2}},$$

$$c_z=\frac{\omega}{\ell}=\frac{k}{\ell}\left(\bar{u}\pm\sqrt{\frac{N^2}{k^2+\ell^2}}\right)=\left(\frac{k}{\ell}\right)\left(\bar{u}\pm\frac{N}{\sqrt{k^2+\ell^2}}\right)=\left(\frac{k}{\ell}\right)c_x$$

The horizontal and vertical group velocities are:

$$c_{gx}=\frac{\partial\omega}{\partial k}=\bar{u}\pm\frac{\partial}{\partial k}\frac{Nk}{\sqrt{k^2+\ell^2}}=\bar{u}\pm\left(\frac{N}{\sqrt{k^2+\ell^2}}-\frac{1}{2}\frac{2Nk^2}{(k^2+\ell^2)^{3/2}}\right)=\bar{u}\pm\frac{N}{\sqrt{k^2+\ell^2}}\left(1-\frac{k^2}{k^2+\ell^2}\right),$$

$$c_{gz}=\frac{\partial\omega}{\partial\ell}=\pm\frac{\partial}{\partial\ell}\frac{Nk}{\sqrt{k^2+\ell^2}}=\pm\frac{1}{2}\frac{-2Nk\ell}{(k^2+\ell^2)^{3/2}}=\pm\frac{N}{\sqrt{k^2+\ell^2}}\left(-\frac{k\ell}{k^2+\ell^2}\right)$$

The horizontal group velocity is always in the same direction as the phase velocity, but the propagating part is less than the phase velocity. As k becomes large (short wavelengths) it approaches zero. As ℓ becomes large the horizontal group and phase velocities become nearly the same, even though both slow. The vertical group velocity is always less than the propagating part of the vertical phase velocity. It is also always directed in the opposite direction. Thus, if waves are excited at the surface so that energy propagates upward, the phase lines will propagate downward, and conversely for waves excited aloft.

Topographic Buoyancy Waves: Consider waves excited by sinusoidal topography, i.e., $h(x) = h_0 \sin kx$, with wavenumber k. If the mountains are not too high, the vertical velocity at the surface is approximately $w(z = 0) \approx U\partial h/\partial x$. These studies are normally formulated in terms of vertical velocity rather than Exner function. Even so the dispersion relation with amplitude and horizontal waveleingth deduced from the surface topography and $\omega = 0$ is a good place to start.

$$k^2\overline{u}^2(k^2+\ell^2)=N^2k^2$$

Which rearranges to:

$$\ell^2=\frac{N^2}{\overline{u}^2}-k^2$$

The wave are sinusoidal when $k^2 < N^2/\overline{u}^2$ and evanescent (decaying exponentially upward) when $k^2 > N^2/\overline{u}^2$. We choose the negative root in order to eliminate the $e^{\ell z}$ part of the solution and keep the solution bounded at large z. The vertical velocity at the surface is $w = k\overline{u}\cos kx$. If we substitute into the zonal momentum equation and continuity equations at the top page 3 and the polarization equation for w in the middle, we can find the amplitudes (U, W, and Π) of the perturbations:

$$\begin{aligned}
&-ik\overline{u}U=-c_p\overline{\theta}(-ik\Pi),\\
&-ikU-i\ell W=0,\\
&(-k^2\overline{u}^2+N^2)W=-c_p\overline{\theta}(-ik)(-i\ell)\Pi,
\end{aligned}$$

which simplify to:

$$\begin{aligned}
&U=-c_p\overline{\theta}\Pi/\overline{u},\\
&W=-(k/\ell)U,\\
&\Pi=[(N^2-k^2\overline{u}^2)/(c_p\overline{\theta}k\ell)]W.
\end{aligned}$$

These relations are useful, for example, to get the Exner function amplitude in terms of the surface boundary condition.

$$\begin{aligned}
\Pi&=[(N^2-k^2\overline{u}^2)/(c_p\overline{\theta}k\ell)](-ikh_0\overline{u})\\
&=-ih_0\overline{u}[(N^2-k^2\overline{u}^2)/(c_p\overline{\theta}\ell)]\\
&=\mp ih_0\overline{u}[(N^2-k^2\overline{u}^2)/(c_p\overline{\theta}\sqrt{N^2/\overline{u}^2-k^2})]\\
&=\mp ih_0\overline{u}^2\sqrt{N^2-k^2\overline{u}^2}/(c_p\overline{\theta})
\end{aligned}$$