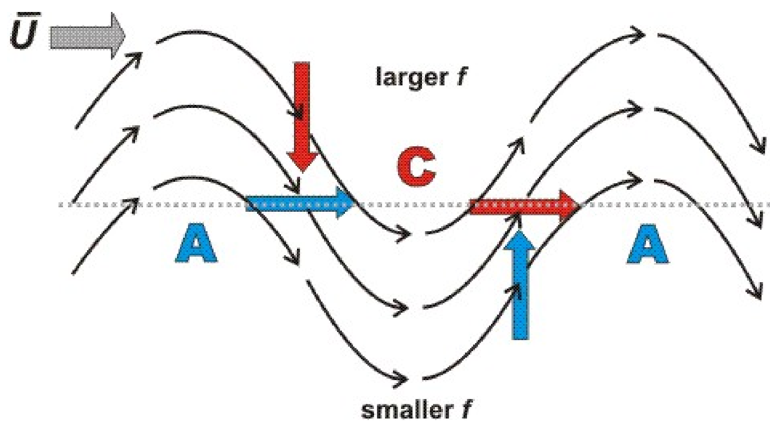
Objective: To study the properties of barotropic Rossby waves.

Reading: CH 5, pp. 161-165

Problems: 5.15 on p. 168

Overview: One-dimensional vorticity advection in a sinusoidal QG, nondivergent flow pattern on a beta plane.



The individual derivative of (conserved) vorticity gives us, $\partial u / \partial t = -\left[\bar{u}\partial u / \partial x + \beta v\right]$. Note that the meridional advection of planetary vorticity always counteracts the zonal advection of relative vorticity. If $\bar{u}\partial u / \partial x$ is larger in magnitude than βv the wave pattern will move eastward with a speed slower than the mean flow. Conversely, if $\bar{u}\partial u / \partial x$ is smaller in magnitude than βv the wave pattern will move slowly westward against the mean flow.

To look at the problem more closely, let's make a shallow-water vorticity equation from the linearized momentum and continuity equations:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} - fv + g\frac{\partial h}{\partial x} = 0 \\
&\frac{\partial v}{\partial t} + \bar{u}\frac{\partial v}{\partial x} + fu + g\frac{\partial h}{\partial y} = 0 \quad , \\
&\frac{\partial h}{\partial t} + \bar{u}\frac{\partial h}{\partial x} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0
\end{aligned}$$

where $\bar{u}$ is a constant and $f = f_0 + \beta y$. We take $-\partial / \partial y$ of the zonal momentum equation and $\partial / \partial x$ of the meridional momentum equation and add them together:

$$\frac{\partial}{\partial t}\left(-\frac{\partial u}{\partial y}\right)+\bar{u}\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)+f_0\frac{\partial v}{\partial y}+\beta v-g\frac{\partial^2 h}{\partial x\partial y}=0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right)+\bar{u}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\qquad +f_0\frac{\partial u}{\partial x}\qquad g\frac{\partial^2 h}{\partial x\partial y}=0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\bar{u}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+f_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\beta v+0=0$$

Substituting from mass continuity:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\bar{u}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-\frac{f_0}{H}\left(\frac{\partial h}{\partial t}+\bar{u}\frac{\partial h}{\partial x}\right)+\beta v+0=0$$

For starters, assume nondivergent flow and neglect the meridional variation of *f* in calculation of wind and vorticity from the mass (*h*) field, QG fashion.

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{\partial}{\partial x}\left(\frac{g}{f_o}\frac{\partial h}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{g}{f_o}\frac{\partial h}{\partial y}\right)=\frac{g}{f_o}\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial x^2}\right),\quad v=\frac{g}{f_o}\frac{\partial h}{\partial x}$$

So

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left[\frac{g}{f_0}\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)\right]\quad -0\quad +\beta\frac{g}{f_0}\frac{\partial h}{\partial x}=0$$

Writing $\phi = gh$ and multiplying by f_0,

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)+\beta\frac{\partial h}{\partial x}=0$$

We could have reached this point by eliminating vertical and meridional structure from the linearized QG vorticity equation. Let's look for one-dimensional propagating waves with no meridional structure. $\phi = A\exp\{-i(\omega t - kx)\}$.

$$-i(\omega-\bar{u}k)(-k^2)+i\beta k=0$$

Or,

$$\omega=\bar{u}k-\frac{\beta}{k}$$

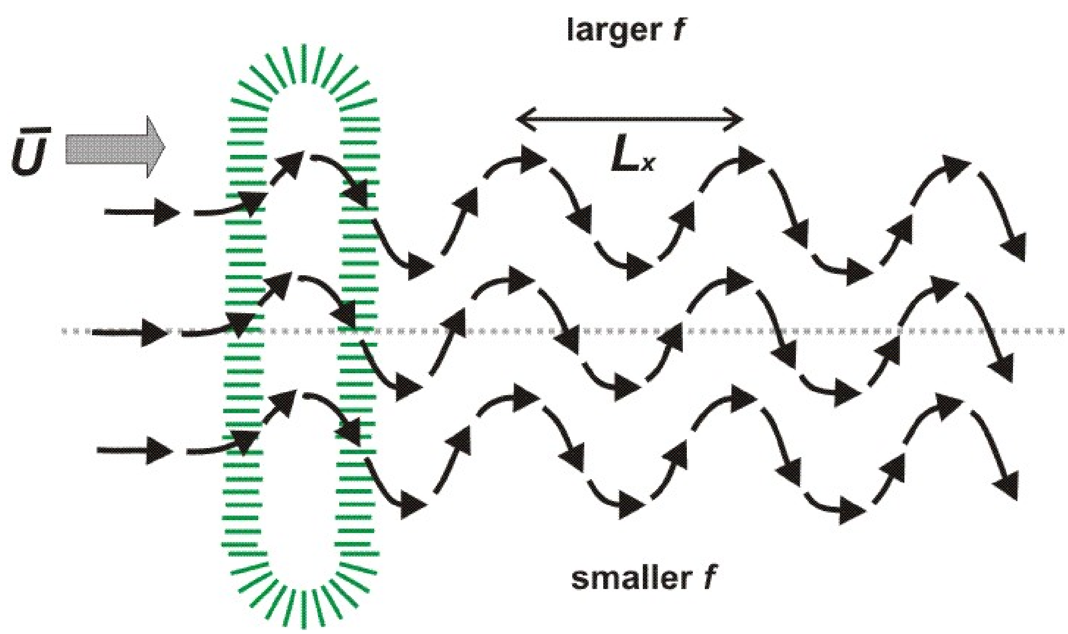
The phase velocity = $c_x=\frac{\omega}{k}=\bar{u}-\frac{\beta}{k^2}$, and the group velocity = $c_{gx}=\frac{\partial\omega}{\partial k}=\bar{u}+\frac{\beta}{k^2}$. Thus, in westerly wind the phase lines move downwind, unless $\bar{u}<\beta/k^2$. What this means is that sufficient long waves (i.e., those with sufficiently small k^2) can retrograde, or propagate upwind against a mean westerly current. For shorter waves the motion is downstream, but slower than the wind in westerly mean flow.

In easterly winds, the phase lines always move downwind faster than the mean flow.

The group velocity is always toward the east relative to the mean flow---opposite to the phase velocity, but with the same magnitude.

Only when $\overline{u} < 0$ (i.e., in easterly mean flow) and $|\overline{u}| > \beta / k^2$ does the energy move westward.

In westerly flow the phase lines are stationary when $\overline{u} = \beta / k^2$. This situation leads to topographic excitation of standing waves with length given by $k = 2\pi / L_x = \sqrt{\beta / \overline{u}}$ or $L_x = 2\pi\sqrt{\overline{u} / \beta}$

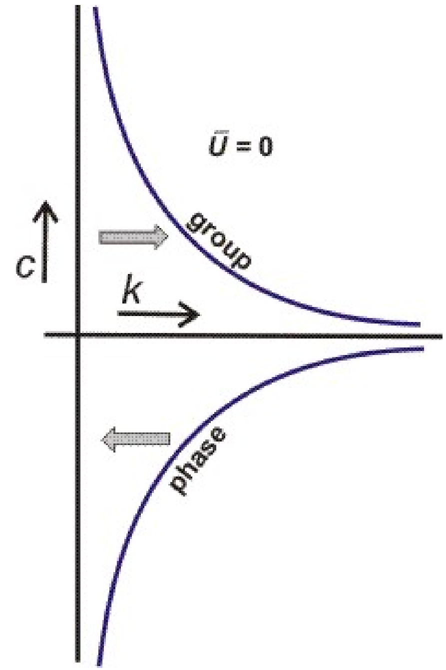


Definitions:

Intrinsic or **Doppler shifted** frequency = $\omega - uk$

Apparent frequency = ω

Dispersion diagram for one-dimensional Rossby waves:



If we now look at two dimensional, but still nondivergent Rossby waves, we need the 2D shallow-water vorticity equation:

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)+\beta\frac{\partial h}{\partial x}=0$$

We assume 2D wave solutions $\phi = A\exp\{-i(\omega t - kx - \ell y)\}$

Substituting into the vorticity equation produces:

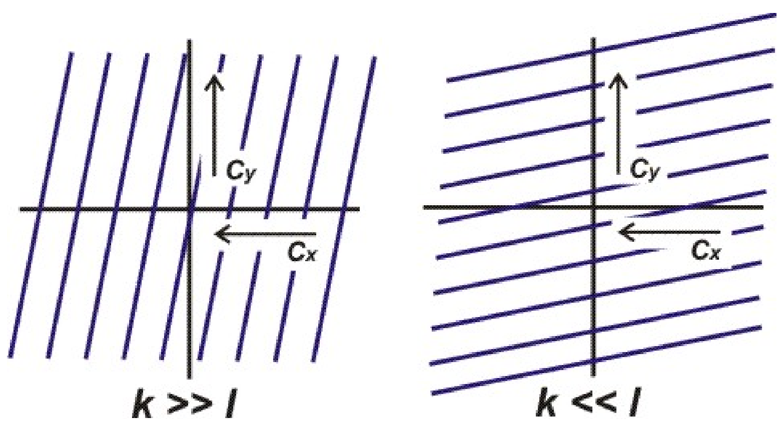
$$-i(\omega-\bar{u}k)(-k^2-\ell^2)+i\beta k=0$$

$$\omega = k\bar{u}-\frac{\beta k}{k^2+\ell^2}$$

The components of the phase velocity are

$$c_x=\frac{\omega}{k}=\bar{u}-\frac{\beta}{k^2+\ell^2}, \qquad c_y=\frac{\omega}{\ell}=\left(\bar{u}-\frac{\beta}{k^2+\ell^2}\right)\frac{k}{\ell}=c_x\frac{k}{\ell}$$

The waves always move westward relative to the flow and may move north or south depending upon the sign of ℓ. When ℓ > 0, $c_y > 0$ because of the proportionality between c_x and c_y.



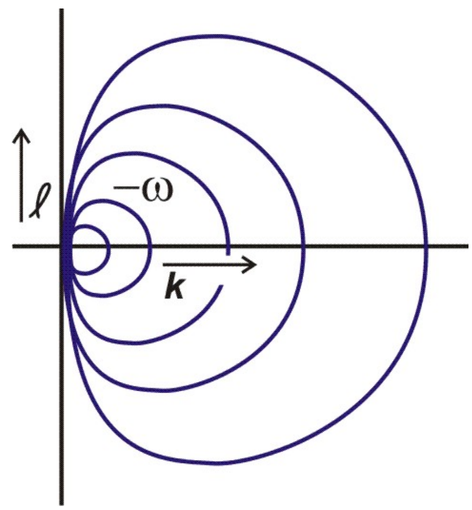
The group velocities are :

$$c_{gx}=\frac{\partial\omega}{\partial k}=\bar{u}-\frac{\beta}{k^2+\ell^2}+\frac{2\beta k^2}{(k^2+\ell^2)^2}$$

$$c_{gy}=\frac{\partial\omega}{\partial \ell}=\frac{2\beta k\ell}{(k^2+\ell^2)^2}$$

The zonal group velocity simplifies to:

$$c_{gx} = \bar{u} - \frac{\beta}{k^2+\ell^2} + \frac{2\beta k^2}{(k^2+\ell^2)^2} = \bar{u} - \frac{\beta}{k^2+\ell^2}\left(\frac{k^2+\ell^2}{k^2+\ell^2} - \frac{2k^2}{k^2+\ell^2}\right)$$
$$= \bar{u} - \frac{\beta}{k^2+\ell^2}\left(\frac{\ell^2-k^2}{k^2+\ell^2}\right) = \bar{u} - \frac{\beta(\ell^2-k^2)}{(k^2+\ell^2)^2} = \bar{u} - \frac{\beta(\ell-k)(\ell+k)}{(k^2+\ell^2)^2}$$



For ℓ > *k*, the zonal group velocity is upstream (relative to the mean flow) like the phase velocity; for ℓ > *k*, the group velocity is downstream. The meridional group velocity can be either northward or southward (dependant upon the sign of ℓ) and is oppositely directed from the meridional phase velocity (apart from the advective contribution to the phase velocity).

For 2-dimenaional divergent Rossby waves we start with the complete shallow-water vorticity equation:

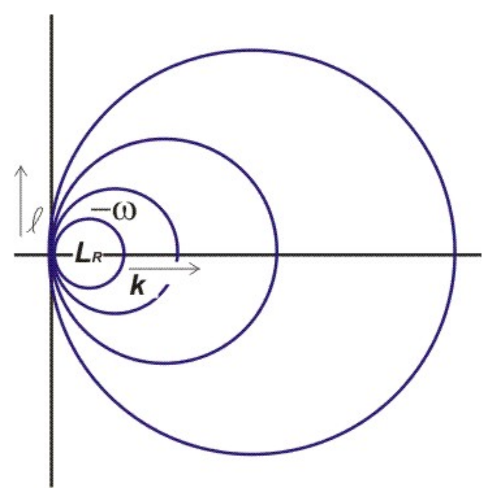
$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left[\frac{g}{f_0}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)\right] - \frac{f_0}{H}\left(\frac{\partial h}{\partial t} + \bar{u}\frac{\partial h}{\partial x}\right) + \beta\frac{g}{f_0}\frac{\partial h}{\partial x} = 0$$

Divide through by g/f_0 to get,

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) - \frac{f_0^2}{gH}\left(\frac{\partial h}{\partial t} + \bar{u}\frac{\partial h}{\partial x}\right) + \beta\frac{\partial h}{\partial x} = 0$$

Write, as before, $h = A\exp\{-i(\omega t - kx - \ell y)\}$ and substitute into the vorticity equation,

$$-i(\omega - \bar{u}k)(-k^2 - \ell^2 - f_0^2/gH) + i\beta k = 0$$
$$\omega = k\bar{u} - \frac{\beta k}{k^2+\ell^2+f_0^2/gH}$$



The phase velocities are:

$$c_x = \frac{\omega}{k} = \bar{u} - \frac{\beta}{k^2+\ell^2+f_0^2/gH},$$
$$c_y = \frac{\omega}{\ell} = \left(\bar{u} - \frac{\beta}{k^2+\ell^2+f_0^2/gH}\right)\frac{k}{\ell} = c_x\frac{k}{\ell}$$

The quantity f_0^2 / gH is the square of the ratio of the Coriolis parameter to the shallow-water phase speed. It's dimensions are (length)-2. The quantity $L_R = \sqrt{gH} / f_0$ is called the Rossby radius of deformation. A typical value is [(9.8 m s-2)(7000 m)]$^{1/2}$/(10^{-4} s^{-1})= 2.63 x 10^6 m = 2620 km. If you re-do this calculation with an internal boundary you get two L_Rs, a big external L_R and a much smaller internal L_R that reflects the slower propagation on an internal density boundary. The group velocities are:

$$
\begin{aligned}
c_{gx} &= \frac{\partial \omega}{\partial k} \\
&= \bar{u} - \frac{\beta}{k^2 + \ell^2 + f_0^2 / gH} + \frac{2\beta k^2}{(k^2 + \ell^2 + f_0^2 / gH)^2} \\
&= \bar{u} - \frac{\beta(\ell^2 + f_0^2 / gH - k^2)}{(k^2 + \ell^2 + f_0^2 / gH)^2} \\
c_{gy} &= \frac{\partial \omega}{\partial \ell} = \frac{2\beta k \ell}{(k^2 + \ell^2 + f_0^2 / gH)^2}
\end{aligned}
$$

Thus, the group velocity changes sign when $k^2 = \ell^2 + f_0^2 / gH$. On the k axis (where $\ell^2 = 0$) the change in sign occurs at $k^2 = L_R^{-2}$. That value also marks the point where the apparent frequency has its largest negative value.

For **Three-Dimesnional QG Rossby Waves** we linearize the QG PV equation:

$$
\frac{\partial}{\partial t}\left[\frac{1}{f_0}\nabla^2\phi + \frac{f_0}{\sigma}\frac{\partial^2\phi}{\partial p^2}\right] + \vec{v}_g \cdot \nabla\left[\frac{1}{f_0}\nabla^2\phi + \frac{f_0}{\sigma}\frac{\partial^2\phi}{\partial p^2} + f_0 + \beta y\right] = 0
$$

To get,

$$
\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left[\frac{1}{f_0}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{f_0}{\sigma}\frac{\partial^2\phi}{\partial p^2}\right] + \beta\frac{1}{f_0}\frac{\partial\phi}{\partial x} = 0
$$

Multiplying by f_0:

$$
\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{f_0^2}{\sigma}\frac{\partial^2\phi}{\partial p^2}\right] + \beta\frac{\partial\phi}{\partial x} = 0
$$

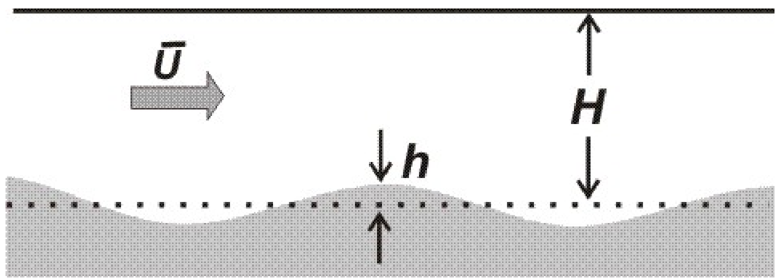
Recall that $\sigma = (\partial\phi_0 / \partial p)(\theta_0^{-1}\partial\theta_0 / \partial p)$ is the QG pressure-coordinate stability. It has units of m^2 s^{-2} mb^{-2} so that f_0^2 / σ has units of mb^2 m^{-2}. Assume solutions, $\phi = A\exp\{-i(\omega t - kx - \ell y - mp)\}$, as previously, but with a pressure-coordinate vertical wavenumber m.

$$
\begin{aligned}
&-i(\omega - \bar{u}k)[-k^2 - \ell^2 - (f_0^2 / \sigma)m^2] + i\beta k = 0 \\
&\omega = k\bar{u} - \frac{\beta k}{k^2 + \ell^2 + (f_0^2 / \sigma)m^2}
\end{aligned}
$$

The group and phase velocities work out as before.

Topographically Forced Rossby Waves:

Apply the vorticity equation to a shallow fluid in one-dimensional flow under a rigid lid and over a sinusoidal bottom.



$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{1}{f_0}\frac{\partial^2\phi}{\partial x^2}\right)+\beta\frac{1}{f_0}\frac{\partial\phi}{\partial x}-f_0\left(\nabla\bullet\vec{v}\right)=0$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{1}{f_0}\frac{\partial^2\phi}{\partial x^2}\right)+\beta\frac{1}{f_0}\frac{\partial\phi}{\partial x}-\frac{f_0}{gH}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\phi_s=0$$

Write the bottom topography as $h = h_o e^{-ikx}$ or $\phi_s = g\, h_o e^{-ikx} = \phi_0 e^{-ikx}$. The waves are steady so that we can assume solutions of the form $\phi = Ae^{-ikx}$.

$$[(0-ik\bar{u})(-k^2)-ik\beta]A=\frac{f_0^2}{gH}(-ik)\bar{u}\phi_0$$

$$(-\bar{u}k^2+\beta)A=\frac{f_0^2}{gH}\bar{u}\phi_0$$

We divide through by $\bar{u}$ and recognize that $\beta/\bar{u}=k_S^2$ is the wavenumber for which one-dimensional, free Rossby waves will be stationary in the given mean flow, so that

$$A(-k^2+\beta/\bar{u})=A(-k^2+k_S^2)=(f_0^2/gH)\phi_0$$

$$A=(f_0^2\phi_0/gH)/(k_S^2-k^2)$$

The horizontal wavenumber is determined by the topography, but the wind may assume reasonable value. For wind speeds such that the (given) horizontal wavenumber is close to that for stationary waves the topographically-forced waves can grow very large. This is one of the effects that set up things like lee waves.