Consider the two-dimensional (x and z), linearized, nonhydrostatic, non-rotational, x-momentum, y-momentum, and buoyancy equations as derived previously using Exner-function/potential temperature thermodynamics:

$$\frac{\partial u'}{\partial t}+\overline{u}\frac{\partial u'}{\partial x}=-c_p\overline{\theta}\frac{\partial \pi'}{\partial x},$$
$$\frac{\partial w'}{\partial t}+\overline{u}\frac{\partial w'}{\partial x}-b'=-c_p\overline{\theta}\frac{\partial \pi'}{\partial z},$$
$$\frac{\partial b'}{\partial t}+\overline{u}\frac{\partial b'}{\partial x}+N^2w'=0.$$

Here the velocity components, potential temperature, $\theta=T\left(p_0/p\right)^{R/c_p}$, and Exner function, $\pi=\left(p/p_0\right)^{R/c_p}$, where *T* is temperature, *p* is pressure, and p_0 is a reference pressure, nominally 1000 kPa; $b'=g\theta'/\overline{\theta}$ is the perturbation buoyancy; and $N^2=(g/\overline{\theta})\partial\overline{\theta}/\partial z$ is the Brunt-Väisälä frequency. The nonlinear continuity equation is:

$$\frac{1}{\rho}\frac{D\rho}{Dt}=-\left(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\right),$$

Defining the density in terms of Exner function and potential temperature,

$$\rho=\frac{p}{RT}=\frac{p_0\pi^{c_p/R}}{R\theta\pi}=\frac{p_0\pi^{(c_p-R)/R}}{R\theta}=\frac{p_0\pi^{c_v/R}}{R\theta}$$

Thus, for adiabatic ($D\theta/Dt=0$) motions:

$$\frac{1}{\rho}\frac{D\rho}{Dt}=\frac{c_v}{R}\frac{1}{\pi}\frac{D\pi}{Dt}.$$

The linearized continuity equation becomes:

$$\frac{c_v}{R}\frac{1}{\overline{\pi}}\left(\frac{\partial \pi'}{\partial t}+\overline{u}\frac{\partial \pi'}{\partial x}+w'\frac{\partial\overline{\pi}}{\partial z}\right)=\frac{c_v}{R}\frac{1}{\overline{\pi}}\left(\frac{\partial \pi'}{\partial t}+\overline{u}\frac{\partial \pi'}{\partial x}-w'\frac{g}{c_p\overline{\theta}}\right)=-\left(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\right)$$
$$\frac{c_v}{R}\frac{1}{\overline{\pi}}\left(\frac{\partial \pi'}{\partial t}+\overline{u}\frac{\partial \pi'}{\partial x}\right)-w'\frac{g}{(c_p/c_v)R\overline{\theta}\overline{\pi}}=-\left(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\right).$$

We recognize that $(c_p/c_v)R\overline{\pi}\overline{\theta}=(c_p/c_v)R\overline{T}=\gamma R\overline{T}=C^2$ is the speed of sound, where $\gamma=c_p/c_v$, and $g/(\gamma RT)=1/\gamma H$

$$\frac{\partial \pi'}{\partial t}+\bar{u}\frac{\partial \pi'}{\partial x}=-\frac{R\bar{\pi}}{c_v}\left[\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}-\frac{w'}{\gamma H}\right],$$

were *H* is the pressure scale height, such that $1/H=g/R\bar{T}=-p^{-1}\partial p/\partial z$ **.** We will need the vertical derivative of mean-flow density:

$$\frac{\partial \bar{\rho}}{\partial z}=\frac{\partial}{\partial z}\frac{\bar{p}}{R\bar{T}}=\frac{1}{R\bar{T}}\frac{\partial \bar{p}}{\partial z}-\frac{\bar{p}}{R\bar{T}^2}\frac{\partial \bar{T}}{\partial z}=\frac{\bar{p}}{R\bar{T}}\left(\frac{1}{\bar{p}}\frac{\partial \bar{p}}{\partial z}-\frac{1}{\bar{T}}\frac{\partial \bar{T}}{\partial z}\right)$$
$$\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial z}==-\frac{1}{H}-\frac{1}{\bar{T}}\frac{\partial \bar{T}}{\partial z},$$

The vertical derivative of mean-flow potential temperature is (remembering that $c_p-c_v=R$ and $\gamma=c_p/c_v$):

$$\frac{\partial \bar{\theta}}{\partial z}=\frac{\partial}{\partial z}\bar{T}\left(\frac{p_o}{\bar{p}}\right)^{R/c_p}=-\bar{T}\frac{R}{c_p}\frac{1}{p}\left(\frac{p_o}{\bar{p}}\right)^{R/c_p}\frac{\partial \bar{p}}{\partial z}+\frac{\partial \bar{T}}{\partial z}\left(\frac{p_o}{\bar{p}}\right)^{R/c_p}=\bar{\theta}\left[-\frac{R}{c_p}\frac{1}{\bar{p}}\frac{\partial \bar{p}}{\partial z}+\frac{1}{\bar{T}}\frac{\partial \bar{T}}{\partial z}\right]$$
$$\frac{1}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z}=\frac{R}{c_p}\frac{1}{H}+\frac{1}{\bar{T}}\frac{\partial T}{\partial z}=-\frac{c_p-c_v}{c_pH}+\frac{1}{\bar{T}}\frac{\partial T}{\partial z}=\left(1-\frac{1}{\gamma}\right)\frac{1}{H}+\frac{1}{\bar{T}}\frac{\partial T}{\partial z}$$

Adding the logarithmic derivatives of density and potential temperature and rearranging,

$$\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial z}+\frac{1}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z}=-\frac{1}{\gamma H}$$
$$\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial z}\equiv-\frac{1}{H_\rho}=-\frac{1}{\gamma H}-\frac{1}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z}.$$

Make a “magic operator” from the vertical momentum and buoyancy equations:

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2 w'-\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)b'=-c_p\bar{\theta}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial \pi'}{\partial z}$$
$$\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2+N^2\right]w'=-c_p\bar{\theta}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial \pi'}{\partial z}$$

Apply to the continuity equation:

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\pi' =$$

$$-\frac{R\overline{\pi}}{c_v}\left\{\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial u'}{\partial x}+\left(\frac{\partial}{\partial z}-\frac{1}{\gamma H}\right)\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]w'\right\} =$$

$$-\frac{R\overline{\pi}}{c_v}\left\{\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial u'}{\partial x}+\left(\frac{\partial}{\partial z}-\frac{1}{\gamma H}\right)\left[-c_p\overline{\theta}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial \pi'}{\partial z}\right]\right\} =$$

$$-\frac{R\overline{\pi}}{c_v}\left\{\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial u'}{\partial x}-c_p\overline{\theta}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \pi'}{\partial z^2}-\left(\frac{1}{\gamma H}-\frac{1}{\overline{\theta}}\frac{\partial \overline{\theta}}{\partial z}\right)\frac{\partial \pi'}{\partial z}\right)\right\}$$

Differentiate again,

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\pi' =$$

$$-\frac{R\overline{\pi}}{c_v}\left\{\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial u'}{\partial x}-c_p\overline{\theta}\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 \pi'}{\partial z^2}-\left(\frac{1}{\gamma H}-\frac{1}{\theta}\frac{\partial \theta}{\partial z}\right)\frac{\partial \pi'}{\partial z}\right)\right\},$$

and substitute from the horizontal momentum equation,

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\pi' =$$

$$\frac{c_pR\overline{\pi}\overline{\theta}}{c_v}\left\{\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial^2 \pi}{\partial x^2}+\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 \pi'}{\partial z^2}-\left(\frac{1}{\gamma H}-\frac{1}{\theta}\frac{\partial \theta}{\partial z}\right)\frac{\partial \pi'}{\partial z}\right)\right\} =$$

$$C^2\left\{\left[\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2+N^2\right]\frac{\partial^2 \pi}{\partial x^2}+\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 \pi'}{\partial z^2}-\left(\frac{1}{\gamma H}-\frac{1}{\theta}\frac{\partial \theta}{\partial z}\right)\frac{\partial \pi'}{\partial z}\right)\right\},$$

which rearranges to:

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^4\pi' - C^2\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2\left\{-\frac{N^2}{C^2}\pi+\left[\frac{\partial^2 \pi}{\partial x^2}+\frac{\partial^2 \pi'}{\partial z^2}-\left(\frac{1}{\gamma H}-\frac{1}{\theta}\frac{\partial \theta}{\partial z}\right)\frac{\partial \pi'}{\partial z}\right]\right\}-C^2N^2\frac{\partial^2 \pi}{\partial x^2}=0\,.$$

Assume solutions of the form $Ae^{\alpha z}e^{i(\omega t-kx-mz)}$, so that $\partial \pi / \partial z=(\alpha-im)\pi$ and $\partial^2 \pi / \partial z^2=(\alpha^2-2i\alpha m-m^2)\pi$.

$$[i(\omega-k\overline{u})]^4$$
$$-C^2[i(\omega-k\overline{u})]^2\left[-\frac{N^2}{C^2}+\left(-k^2-(m^2+2im\alpha-\alpha^2)-\left(\frac{1}{\gamma H}-\frac{1}{\theta}\frac{\partial \theta}{\partial z}\right)(\alpha-im)\right)\right]$$
$$+C^2N^2k^2=0$$

The real and imaginary terms must vanish separately. Selecting the imaginary terms and assuming that m ≠ 0,

$$-2im\alpha + im\left(\frac{1}{\gamma H} - \frac{1}{\theta}\frac{\partial\theta}{\partial z}\right) = 0,$$

$$\alpha = \frac{1}{2}\left(\frac{1}{\gamma H} - \frac{1}{\theta}\frac{\partial\theta}{\partial z}\right)$$

We recognize that,

$$\frac{N^2}{C^2} = \frac{1}{\gamma RT}\frac{g}{\theta}\frac{\partial\theta}{\partial t} = \frac{g}{\gamma RT}\frac{1}{\theta}\frac{\partial\theta}{\partial t} = \frac{1}{\gamma H}\frac{1}{\theta}\frac{\partial\theta}{\partial t}$$

Substituting into the real terms,

$$[(\omega - k\overline{u})]^4 - [i(\omega - k\overline{u})]^2 C^2 \left\{-\frac{1}{\gamma H}\frac{1}{\theta}\frac{\partial\theta}{\partial z} + \left[\left(-k^2 - m^2 + \frac{1}{4}\left(\frac{1}{\gamma H} - \frac{1}{\theta}\frac{\partial\theta}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{1}{\gamma H} - \frac{1}{\theta}\frac{\partial\theta}{\partial z}\right)\left(\frac{1}{\gamma H} - \frac{1}{\theta}\frac{\partial\theta}{\partial z}\right)\right)\right]\right\} + C^2N^2k^2 = 0$$

$$[(\omega - k\overline{u})]^4 - [(\omega - k\overline{u})]^2 C^2 \left\{\frac{1}{\gamma H}\frac{1}{\theta}\frac{\partial\theta}{\partial z} + k^2 + m^2 + \frac{1}{4}\left(\frac{1}{\gamma H} - \frac{1}{\theta}\frac{\partial\theta}{\partial z}\right)^2\right\} + C^2N^2k^2 =$$

$$[(\omega - k\overline{u})]^4 - [(\omega - k\overline{u})]^2 C^2 \left\{\frac{1}{\gamma H}\frac{1}{\theta}\frac{\partial\theta}{\partial z} + k^2 + m^2 + \frac{1}{4}\left[\left(\frac{1}{\gamma H}\right)^2 - 2\frac{1}{\gamma H}\frac{1}{\theta}\frac{\partial\theta}{\partial z} + \left(\frac{1}{\theta}\frac{\partial\theta}{\partial z}\right)^2\right]\right\} + C^2N^2k^2 =$$

$$[(\omega - k\overline{u})]^4 - [(\omega - k\overline{u})]^2 C^2 \left\{k^2 + m^2 + \frac{1}{4}\left(\frac{1}{\gamma H} + \frac{1}{\theta}\frac{\partial\theta}{\partial z}\right)^2\right\} + C^2N^2k^2 =$$

$$[(\omega - k\overline{u})]^4 - [(\omega - k\overline{u})]^2 C^2 \left(k^2 + m^2 + \frac{1}{4}\frac{1}{H_\rho^2}\right) + C^2N^2k^2$$

The first two terms describe high-frequency waves:

$$[(\omega - k\overline{u})]^4 - [(\omega - k\overline{u})]^2 C^2 \left(k^2 + m^2 + \frac{1}{4H_\rho^2}\right) = 0$$

$$[(\omega - k\overline{u})]^2 = C^2 \left(k^2 + m^2 + \frac{1}{4H_\rho^2}\right),$$

$$\omega = k\overline{u} \pm C\sqrt{k^2 + m^2 + \frac{1}{4H_\rho^2}}.$$

For short waves, $(k^2, m^2) \gg H_*^{-1}$, these are the 2-dimensional version of the same sound waves we derived earlier $\omega = k\bar{u} \pm C\sqrt{k^2+m^2}$. For long waves, $(k^2, m^2) \sim H_\rho^{-1}$, that is for wave lengths comparable with twice the modified scale height, the frequency approaches $C/2H_*$, called the ***Acoustic Cutoff Frequency.***

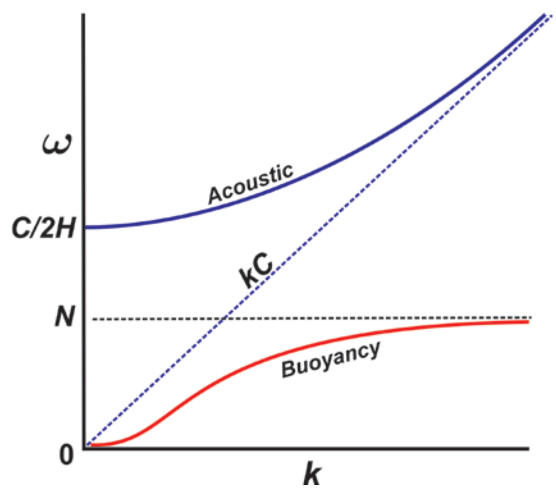
The last two terms define non-rotational internal gravity waves:

$$-[(\omega - k\bar{u})]^2 C^2 \left(k^2 + m^2 + \frac{1}{4H_\rho^2} \right) + C^2 N^2 k^2 = 0,$$

$$[(\omega - k\bar{u})]^2 \left(k^2 + m^2 + \frac{1}{4H_\rho^2} \right) = N^2 k^2,$$

$$\omega = k\bar{u} \pm \sqrt{\frac{N^2 k^2}{k^2 + m^2 + \frac{1}{4H_\rho^2}}}.$$

As before, the frequencies of horizontally short waves (large k or small m) approach $\pm N$. The frequencies of horizontally long wave (small k or large m) approach zero. If we had included rotation, the frequency limit horizontally long wave would be $\pm f$.



The vertical scale for wave amplitude can be rewritten,

$$\alpha = \frac{1}{2}\left(\frac{1}{\gamma H} - \frac{1}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z} \right) = \frac{1}{2}\left(\frac{g}{C^2} - \frac{N^2}{g} \right) = \frac{1}{2H_\rho} - \frac{N^2}{g}$$