Supplemental Material: Matrix Solution for Rotational Gravity Waves:

Consider the equations for rotational shallow-water gravity waves:

$$\left(\frac{\partial}{\partial t}-\bar{u}\frac{\partial}{\partial x}\right)u-fv+g\frac{\partial h}{\partial x}=0,$$
$$\left(\frac{\partial}{\partial t}-\bar{u}\frac{\partial}{\partial x}\right)v+fu+g\frac{\partial h}{\partial y}=0,$$
$$\left(\frac{\partial}{\partial t}-\bar{u}\frac{\partial}{\partial x}\right)h+H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)=0.$$

Assume a solution of the form:

$$\begin{bmatrix} u \\ v \\ h \end{bmatrix}=\begin{bmatrix} U \\ V \\ A \end{bmatrix}e^{i(\omega t-kx-\ell y).}$$

Substitute into the governing equations and reorder the terms,

$$i(\omega-k\bar{u})u-fv-ikgh=0,$$
$$fu+i(\omega-k\bar{u})v-i\ell gh=0,$$
$$-ikHu-i\ell Hv+i(\omega-k\bar{u})=0.$$

This expression may be rewritten in matrix form:

$$\begin{bmatrix} i(\omega-k\bar{u}) & -f & -ikg \\ f & i(\omega-k\bar{u}) & -i\ell g \\ -ikH & -i\ell H & i(\omega-k\bar{u}) \end{bmatrix}\begin{bmatrix} u \\ v \\ h \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Nonzero u, v, and h are possible only if the determinant of the matrix on the left is zero:

$$\begin{vmatrix} i(\omega-k\bar{u}) & -f & -ikg \\ f & i(\omega-k\bar{u}) & -i\ell g \\ -ikH & -i\ell H & i(\omega-k\bar{u}) \end{vmatrix}=0$$

Expanding by minors:

$$i(\omega-k\bar{u})\begin{vmatrix} i(\omega-k\bar{u}) & -i\ell g \\ -i\ell H & i(\omega-k\bar{u}) \end{vmatrix}+f\begin{vmatrix} f & -i\ell g \\ -ikH & i(\omega-k\bar{u}) \end{vmatrix}-ikg\begin{vmatrix} f & i(\omega-k\bar{u}) \\ -ikH & -i\ell H \end{vmatrix}=0\,.$$

Again,

$$i(\omega-k\bar{u})[-(\omega-k\bar{u})^2+gH\ell^2]+f[if(\omega-k\bar{u})+k\ell gH]-ikg[-fi\ell H+(\omega-k\bar{u})kH]=0.$$

Expanding the second and third brackets,

$$i(\omega - k\overline{u})[-(\omega - k\overline{u})^2 + gH\ell^2] + if^2(\omega - k\overline{u}) + fk\ell gH - fk\ell gH + i(\omega - k\overline{u})k^2 gH = 0$$

Canceling the $fk\ell gH$ terms and dividing through by $i(\omega - k\overline{u})$ produces,

$$-(\omega - k\overline{u})^2 + gH\ell^2 + f^2 + gHk^2 = 0,$$

Which yields:

$$\omega - k\overline{u} = \pm\sqrt{gH(k^2 + \ell^2) + f^2}\ .$$

This is the same result that we obtained before, but here by a process that is more mechanical and boring. It is, however, used commonly.