Final Exam: Monday, 23 April 2018 in AHC5-357, 12 Noon-2PM

The final exam will have 20 multiple-choice questions and ten long-answers, problems or draw and explain. Of the latter you will need to answer 6. It will be worth 1/3 of tour grade. You will have the entire two hours to complete the exam.

1. Know that "diagnostic" equations do not contain explicit time derivatives. Examples include the gas law and the omega equation. Prognostic equations do contain at least one explicit time derivative. Example include the vorticity and momentum equations.

2. Know that prognostic equations cast in "initial value form (IVF)" contain only one first derivative with respect to time on the left hand side and only variables themselves or their spatial parial derivatives on the right.

3. Understand the perturbation method and be able to apply it to linearized typical meteorological equations.

4. A periodic function returns to the same value every even multiple of its time period of *T* or spatial wavelength *L*: $f(t + T) = f(t)$ or $g(x + L) = g(x)$.

5. Understand exponential representation of trigonometric functions (Euler's equation $e^{i\theta} = \cos\theta + i\sin\theta$ or $e^{-i\theta} = \cos\theta - i\sin\theta$) and be able to it apply to linearized meteorological equations.

6. Know the definitions of frequency $\omega = 2\pi / T$ and wavenumbers in the x and y directions $k = 2\pi / L_x$ and $\ell = 2\pi / L_y$, where *T* is the wave period and L_x and L_y are the wavelengths in the *x* and *y* directions. Be able to use these definitions to describe a propagating sinusoidal wave.

7. Understand representation of propagating waves with <u>Euler's equation</u> using a <u>complex amplitude</u> $A = B + Ci$:

$$\begin{aligned}\mathrm{Re}\left\{Ae^{i(\omega t-kx)}\right\} &= \mathrm{Re}\left\{(B-Ci)(\cos(\omega t-kx)+i\sin(\omega t-kx)\right\}\\ &= \mathrm{Re}\left\{B\cos(\omega t-kx)+C\sin(\omega t-kx)+i([-C\cos(\omega t-kx)+B\sin(\omega t-kx)]\right\}\\ &= B\cos(\omega t-kx)+C\sin(\omega t-kx)\end{aligned}$$

8. Know the definitions of phase velocity $(c_x = \omega / k, c_y = \omega / \ell)$ and group velocity $(c_{gx} = \partial\omega / \partial k, c_{gy} = \partial\omega / \partial\ell)$. Understand the meaning if each and why they may be different.

9. Given a dispersion relation, $\omega = g(k, \ell)$ be able to calculate the phase and group velocities.

10. Know that in <u>dispersive</u> waves the phase velocity is a variable function of the wavenumber(s). Be able to recognize a dispersive wave from its dispersion relation or phase velocity. A nondispersive waveform simply translates with wave speed *c* with no distortion, such that *f*(*x*) =

$f(x - ct)$. In a dispersive wave, each Fourier component (see below) has its own phase speed so that the waveform changes shape as well as location with time.

11. Know what a Fourier series is and how it can be used to represent any periodic function.

$$f(x) = \sum_{n=1}^{\infty}\left(B_n \cos\frac{2\pi nx}{L} + C_n \sin\frac{2\pi nx}{L} \right)$$

12. Where B_n and C_n are determined from the function to be represented by integrating it times the cosine or sine over the interval $0 \le x \le L$. Fourier series can also be represented using Euler's formula with a complex amplitude $A_n = B_n - iC_n$.

13. Understand why we often analyze meteorological systems using only a single Fourier component at a time.

14. Understand how energy or wave packets move with the group velocity whereas the individual wave crests and troughs move with the phase velocity. Be able to explain these phenomena using a sketch of a wave packet that moves with the group velocity and modulates a higher frequency carrier that moves with the phase velocity.

15. Given the linearized governing equations, be able to derive the dispersion relation for one-dimensional, acoustic waves.

16. Given the linearized governing equations, be able to derive the dispersion relation for one-dimensional, shallow water gravity waves without background rotation.

17. Given one or more governing equations (for exmple $\overline{\rho}(\partial / \partial t + \overline{u}\partial / \partial x)u' = -\partial p / \partial x$ and solutions for u and h of the form $p = Pe^{ik(x-ct)}$ and $u' = Ue^{ik(x-ct)}$), be able to express one amplitude U or P, in terms of the other

18. Understand, be able to explain in words, and be able to derive the way that uniform background rotation (i.e. the Coriolis force) changes the dispersion relation and propagation of shallow-water gravity waves.

19. Understand and be able to explain the dispersion relations for rotational and irrotational inertia-gravity waves in a stratified atmosphere and be able to interpret them.

20. Know the definitions and significances of the terms buoyancy, $b = g\theta' / \theta_0$, and the square of the Brunt-Väisälä (buoyancy) frequency, $N^2 = (g / \theta_0)(\partial\theta_0 / \partial z)$.

21. Know generally how to use Exner function $\pi = (p / p_0)^{R/c_p}$ and potential temperature $\theta = T(p_0 / p)^{R/c_p}$ as the thermodynamic variables in the meteorological form of the Navier-Stokes equations.

22. Understand and be able to explain in words what happens in a two layer model in which the upper layer has a free surface and depth H_1. The lower layer has depth H_2, rests on a flat bottom and the relative density contrast is $\sigma = (\rho_2 - \rho_1)/\rho_2$. The solutions are external gravity waves with speed $C_E = \pm\sqrt{g(H_1 + H_2)}$ and slower moving internal gravity waves with speed $C_E = \pm\sqrt{gH_1H_2/(H_1 + H_2)}$. Be able to explain physically how they arise.

23. Know that on a "beta-plane" we represent the Coriolis parameter $f = 2\Omega \sin\varphi \simeq f_0 + \beta(y - y_0)$. On a *f*-plane the Coriolis parameter is simply constant at f_0. On a middle-latitude β-plane y_0 generally corresponds to 45° latitude. On an equatorial β-plane, it corresponds to 0° such that $f_0 = 0$ and $f = \beta y$.

24. Know the definitions of Rossby radius of deformation in the shallow-water formulation $\lambda^{-2} = gH/f_0^2$ and in a continuously stratified atmosphere $\lambda^{-2} = \sigma/f_0^2$ (i.e $R_R = \sqrt{gH}/f_0$ or $R_R = \sigma^{1/2}/f_0$) and know its significance for rotational gravity waves and divergent Rossby waves. Do not confuse the Rossby radius with the Rossby number *U/fL.*

25. Be able to explain inertial instability qualitatively. If the magnitude of the meridional anticyclonic shear of the balanced zonal flow is > *f*, then air parcels displaced to the north move too fast to remain in geostrophic balance and those displaced to the south move too slowly. In either case they accelerate away from their initial latitude. If the shear is cyclonic, or anticyclonic with magnitude < *f,* the displaced parcels will undergo stable inertia oscillations around a center that moves with the balanced mean flow.

26. Understand and be able to explain how large-scale (i.e. Quasigeostrophic) dynamics induce rising motion through:

    a. Acceleration of upper tropospheric ageostrophic wind

    b. Upper tropospheric potential vorticity advection

    c. Positive vorticity advection as forcing for the omega equation

27. Be able to explain qualitatively the reason why the phase of a sinusoidal, balanced vorticity-conserving disturbance on a middle latitude-latitude beta plane will propagate westward in still air.

28. Starting with the vorticity equation, be able to derive the dispersion relation for a quasigeostrophic, barotropic, nondivergent one-dimensional Rossby waves on a middle latitude beta plane.

29. Be able to deduce the wavenumber *k*, for which a one-dimensional nondivergent Rossby wave will be stationary, ω = 0, in a uniform zonal flow.

30. Know how, and why, the dynamics and dispersion relations of two-dimensional and/or divergent Rossby waves differ from the one-dimensional system described in item 28.

31. Be able to calculate and describe group and phase velocities from the dispersion relations for all kinds of Rossby waves that we have analyzed.

32. Know how the Rossby radius of deformation and Rossby-wave cutoff frequency are related for one dimensional, divergent Rossby waves, and be able to derive this relation.

33. Be able to draw and explain the set-up of the two-layer model of quasigeostrophic baroclinic instability, including representation of the vertical velocity, levels at which the vorticity and thermodynamic energy equations are applied, and values of ω at the Earth's surface, mid troposphere, and top of the atmosphere.

34. Know that when the dispersion relation stems from solution of quadratic equation and the quantity under the radical is negative, the frequency contains ± an imaginary part so that $e^{i(\omega_r \pm i\omega i)t} = e^{\mp\omega_i t} e^{i\omega_r t}$. The positive exponential in the first factor will grow exponentially with e-folding time ω_i, leading to instability, as the wave oscillates with frequency ω_r due to the second factor.

35. Be able to describe in words generally how Norman Phillips 2-layer analysis of the vorticity and thermodymanic energy equations leads to a dispersion relation for Rossby waves in a vertically sheared flow. Be able to show how the complex frequency (phase speed) describes growing waves.

36. Understand the finite difference representation of the pressure-coordinate vertical velocity, ω, and its vertical derivatives and the partitioning of the mean flow and perturbation streamfunction into vertical mean and thermal wind components.

37. Using the dispersion relation for Phillips formulation of baroclinic instability,

$$c = U_M + \frac{-\beta(k^2+\lambda^2) \pm \sqrt{\beta^2\lambda^4 - U_T^2 k^4 (4\lambda^4 - k^4)}}{k^2(2\lambda^2 + k^2)},$$

Be able to describe the two special cases where β = 0 and $U_T = 0$.

38. Be able to determine the wavelength with fastest growth by finding the value of *k* that corresponds to the positive maximum of the argument under the radical, $\beta^2\lambda^4 - U_T^2 k^4 (4\lambda^4 - k^4)$.

39. Be able to interpret and label a sketch of the the stability boundary

40. Be able to explain the role of β in stabilizing baroclinic waves and how this conclusion is significant to the general circulation.

41. Be able to explain the shortwave cutoff for $k^2 \geq 2\lambda^2$

42. Know that Poisson's equation has the form $\nabla^2\phi = f(x,y)$. Meteorological equations such as the omega and geopotential tendency equations have this form, albeit with non-constant coefficients multiplying some of the partial derivatives. If the equation is homogeneous, i.e. $\nabla^2\phi = 0$, it is called Laplace's equation. Standard reliable mathematical and numerical techniques exist to solve both P's and L's equations.

43. Using the algebraic expression for the stability boundary $\left(\frac{k^2}{2\lambda^2}\right)^2 = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{\beta^2}{4U_T^2\lambda^4}}\right]$ for the two-layer baroclinic instability problem, be able to sketch and the stability boundary for baroclinic waves. Be sure to include the minimum shear needed for instability, the wavenumber where the instability first appears, and description of the behavior of long and short waves, that is small and large *k*.

44. Be able to show that the integral of the partial derivative $\overline{\partial F / \partial x}$ of any periodic function, such as $F(x+L) = F(L)$, over a complete wavelength (or period for a function of time), is zero,

$$\overline{\frac{\partial F}{\partial x}} = \int_0^L \frac{\partial F}{\partial x} dx = F(x)\Big|_0^L = F(L) - F(0) = 0$$

45. Be able to describe the typical distribution of latitudinally averaged zonal winds on the Earth.

46. Know that surface friction in the tropical easterlies (Trade Winds) is a source of westerly and angular momentum while friction in the middle latitude westerlies is a sink. In a fixed reference frame, air moving with the easterlies rotates about the Earth's axis more slowly than the planet itself; whereas air moving with the westerlies rotates about the axis faster than the planet.

47. Since sunlight falls more directly on the tropics (mainly the tropical seas), low latitudes are a source of latent and sensible heat to the atmosphere. On the other hand, IR radiation emitted from the surface and from the atmosphere is more nearly constant with latitude. Consequently, atmospheric motions redistribute both heat and westerly momentum from the tropics to middle and high latitudes.

48. Know that in low-latitudes the redistribution is accomplished by the zonally symmetric Hadley Circulation, and in middle latitudes it is accomplished primarily by transient eddies (i.e. unstable baroclinic waves).

49. Know the definitions of "thermally direct" and thermally indirect" and that the Hadley and Polar cells are examples of the former, while the Farrel cell is an example of the latter.

50. Know that addition of westerly angular momentum (as by friction in the Trades) leads to equatorward motion away from the axis of rotation toward places where the background angular momentum is larger. Addition of easterly momentum (i.e. destruction of westerly momentum, as by friction in the middle-latitude Westerlies) leads to poleward motion toward

regions of lower planetary angular momentum. Similarly, heating that increases potential temperature leads to rising motion toward the upper atmosphere where background potential temperature is higher; and cooling leads to sinking toward the lower atmosphere where potential temperature is lower.

51. Starting with the zonally averaged zonal momentum and thermodynamic energy equations, geostrophic balance, and mass continuity ensured by defining the zonal and pressure coordinate vertical velocities in terms of a streamfunction,

$$\frac{\partial \overline{u}}{\partial t}+\overline{v}\left(\frac{\partial \overline{u}}{\partial y}-f\right)+\omega\frac{\partial \overline{u}}{\partial p}=M,\ \frac{\partial}{\partial t}\left(\frac{\partial \overline{\phi}}{\partial p}\right)+\overline{v}\frac{\partial}{\partial y}\left(\frac{\partial \overline{\phi}}{\partial p}\right)-\overline{\omega}\sigma=-q,$$
$$f\overline{u}=-\frac{\partial \overline{\phi}}{\partial y},\ \overline{u}=-\frac{\partial \psi}{\partial p},\ \overline{\omega}=\frac{\partial \psi}{\partial y}$$

Understand how to derive the Sawyer-Eliassen Equation for the mean meridional circulation, assuming that the coefficients are constant with latitude and pressure (which they are not in Nature).

$$f\left(\frac{\partial \overline{u}}{\partial y}-f\right)\frac{\partial^2 \psi}{\partial p^2}-2fS\frac{\partial^2 \psi}{\partial p \partial y}+\sigma\frac{\partial^2 \psi}{\partial y^2}=-\frac{\partial (fM)}{\partial p}+\frac{\partial q}{\partial y}$$

52. Be able to sketch the structure of the mean meridional circulation showing the three cells and explain (in terms of the Sawyer Eliassen Equation solutions and forcing) how surface friction, condensational heating in the ITC, momentum and heat transport by middle latitude eddies, and IR radiative cooling force the Hadley, Farrel, and Polar cells.

53. Know what an equatorial β-plane is, i.e., $f = \beta y$. Reference pp. 394-400 in Holton.

54. Know how to use l'Hopital's Rule to deal with the apparent singularity in the geostrophic wind relation $u=-(\partial\phi/\partial y)/f=-(\partial\phi/\partial y)/\beta y=-(\partial^2\phi/\partial y^2)/\beta$ when y= 0 at the equator

55. Understand generally how the momentum and continuity equations combine to produce a single equation for the meridional structure of $\Phi(y')$ on an Equatorial Beta Plane:

$$\frac{\partial^2 \Phi}{\partial y'^2}+\left(\omega'^2-k'^2+\frac{k'}{\omega}-y'^2\right)\Phi=0$$

Where the inverse time and space scales are $\sqrt{C\beta}$ and $\sqrt{\beta/C}$, such that $y'=y(\beta/C)^{1/2}$ is the nondimensional meridional coordinate. By assuming solutions of the form $\Phi(y')=F(y')\exp\{-y'^2/2\}$ and canceling some terms the meridional structure equation becomes,

$$\frac{d^2F(y')}{dy'^2} - 2y' + \left(\omega'^2 - k'^2 + \frac{k'}{\omega} - 1\right)F(y' = \frac{d^2F(y')}{dy'^2} - 2y'\frac{dF(y')}{dy'} + 2\lambda F(y') = 0$$

For λ = 0, 1, 2, 3, …, $F(y') = H_n(y')$. These solutions are Hermite polynomials. If λ is not a positive integer the solutions are unbounded as *y′* becomes large.

56. Know that the "quantum condition" produces a cubic dispersion equation relation for $\lambda = 0$ that can be factored to give an eastward propagating inertia-gravity wave and a westward propagating Rossby-inertia wave and an eastward propagating Kelvin wave that moves with the gravity-wave speed. Somewhat similar waves result for n = 1, 2, … , which produce eastward and westward propagating inertia-gravity waves and an eastward (slowly) propagating equatorial Rossby wave.

57. Know that, as described above, the equatorial β-plane governing equations also allow the special case of Equatorial Kelvin Wave. It has the following properties:

    a. The horizontal motion is entirely in the zonal direction with identically zero meridional velocity.

    b. It propagates eastward with the shallow water gravity-wave speed

    c. It has Gaussian meridional structure where the "standard deviation" in the exponential function is $\sqrt{C/\beta}$

    d. The perturbation flow is in geostrophic balance, eastward in geopotential highs and westward in lows so that there is convergence ahead of the highs and divergence ahead of the lows, consistent with gravity-wave propagation speed.

    <u>You should be able to derive these properties by substituting *v* = 0 into the complete Navier-stokes equations on an equatorial beta plane.</u>

58. Know that internal Kelvin waves that move more slowly than shallow-water waves we discussed in class explain the atmospheric Madden-Julian Oscillation. In the ocean, onset of warm-phase ENSO near and to the east of the Dateline is due to even more slowly propagating internal Kelvin waves propagating on the oceanic thermocline. They are excited by transient westerly equatorial winds.

59. Know and understand the necessary criterion, $(\partial/\partial y)(\zeta + f)$ changes sign, for barotropic instability.