

3.1 Functions

A **relation** is a set of ordered pairs (x, y) .

Example: The set $\{(1,a), (1, b), (2,b), (3,c), (3, a), (4,a)\}$ is a relation

A **function** is a relation (so, it is the set of ordered pairs) that does not contain two pairs with the same first component. Sometimes we say that a function is a rule (correspondence) that assigns to each element of one set, X , one and only one element of another set, Y . The elements of the set X are often called **inputs** and the elements of the set Y are called **outputs**.

A function can be visualized as a machine, that takes x as an input and returns y as an output.

The **domain of a function** is the set of all first components, x , in the ordered pairs.

The **range of a function** is the set of all second components, y , in the ordered pairs.

We will deal with functions for which both domain and the range are the set (or subset) of real numbers

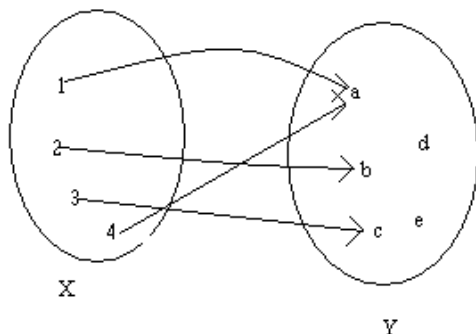
A function can be defined by:

- (i) Set of **ordered pairs**

Example: $\{(1,a), (2,b), (3,c), (4,a)\}$ is a function, since there are no two pairs with the same first component. The domain is then the set $\{1,2,3,4\}$ and the range is the set $\{a,b,c\}$

Example: $\{(1,a), (2,b), (1,c), (4,a)\}$ is not a function, since there are two pairs with the first component 1

- (ii) **Diagram** which shows how the elements of two sets are paired.



1 is paired with a , so $(1,a)$ belongs to function

The domain is the set $X = \{1,2,3,4\}$ and the range is the subset $\{a,b,c\}$ of Y .

- (iii) A **formula or equation** which specifies how the y is obtained when x is given.

Example: $y = 3x + 2$ defines a function since we can easily find the output y when the input, x , is given. If x is 1 then $y = 3(1) + 2 = 5$, for example. So every x has a corresponding y value.

We say that a function is **given explicitly** by an equation, if the equation is of the form

$y =$ (formula that contains only x variable). We say that a function is given **implicitly**, if it is given by an equation that is not in the above form. For example, $y = 3x + 2$ is a function given in an explicit form, and $3x - y + 2 = 0$ is the same function given in an implicit form.

When we use a formula to define a function we often give a name to a function ($f, g, h, \text{etc.}$) and use a special notation for the output, y . If x is the input, then we denote the output by $f(x)$ (read “ f of x ”).

Caution: $f(x)$ is not a multiplication of f and x . It is an entity that can't be split. $f(x)$ denotes output that corresponds to the input x .

For example, instead of writing $y = 3x + 2$, we often write $f(x) = 3x + 2$. With this notation $f(1)$ denotes the output (y) that correspond to the input 1, that is $f(1) = 3 \cdot (1) + 2 = 5$.

If we say that $f(4) = -5$, that means that when the input is 4, the output is -5. This also means that the function f contains a pair $(4, -5)$

When a function f is given by a formula and its domain is not given, then it is assumed that the domain is the largest set of real numbers for which $f(x)$ can be computed and is a real number. An equation defines a function if it can be solved for y and the solution is unique.

Example: $3x + 2y = 4$ is a function, because we can solve it for y , $y = (-3/2)x + 2$ and the solution is unique. The equation $x^2 + y^2 = 1$ is not a function, because when we solve it for y , we get $y = \pm\sqrt{1 - x^2}$, two solutions.

(iv) The **graph**, which is the set of all pairs (x, y) that are plotted in the coordinate system.

The graph is the set of all points $(x, f(x))$, where x belongs to the domain of a function f .

A point (a, b) is on the graph of a function f , if and only if $b = f(a)$

A graph represents a function, if it passes the **Vertical Line Test**, which says that *If any vertical line crosses the graph at most at one point, then the graph is that of a function.*

The domain of a function given by a graph is the set of all x -coordinates of the points on the graph. **The range** is the set of all y -coordinates of the points on the graph

Example: Find $f(0)$, $f(1)$, $f(-2)$, $f(-x)$, $-f(x)$, $f(x+1)$, $f(2x)$, $f(x+h)$ for $f(x) = 3x^2 - 2x - 4$

$$f(0) = 3(0)^2 - 2(0) - 4 = -4$$

$$f(1) = 3(1)^2 - 2(1) - 4 = -3$$

$$f(-2) = 3(-2)^2 - 2(-2) - 4 = 12$$

$$f(-x) = 3(-x)^2 - 2(-x) - 4 = 3x^2 + 2x - 4$$

$$f(x+1) = 3(x+1)^2 - 2(x+1) - 4 = 3(x^2 + 2x + 1) - 2x - 2 - 4 = 3x^2 + 6x + 3 - 2x - 6 = 3x^2 + 4x - 3$$

$$f(2x) = 3(2x)^2 - 2(2x) - 4 = 3(4x^2) - 4x - 4 = 12x^2 - 4x - 4$$

$$f(x+h) = 3(x+h)^2 - 2(x+h) - 4 = 3(x^2 + 2xh + h^2) - 2x - 2h - 4 = 3x^2 + 6xh + 3h^2 - 2x - 2h - 4$$

$$-f(x) = -(3x^2 - 2x - 4) = -3x^2 + 2x + 4$$

Example : Find $\frac{f(x+h)-f(x)}{h}$ (**the difference quotient** for f) for $f(x) = 3x^2 - 2x - 4$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(3(x+h)^2 - 2(x+h) - 4) - (3x^2 - 2x - 4)}{h} = \frac{3(x^2 + 2xh + h^2) - 2x - 2h - 4 - 3x^2 + 2x + 4}{h} \\ &= \frac{3x^2 + 6xh + 3h^2 - 2h - 3x^2}{h} = \frac{6xh + 3h^2 - 2h}{h} = \frac{h(6x + 3h - 2)}{h} = 6x + 3h - 2 \end{aligned}$$

The average rate of change of a function f from a to b is defined as $\frac{f(b)-f(a)}{b-a}$

Example : Find the average rate of change of $f(x) = -x^3 + 3$ from 0 to 2

Here $a = 0$ and $b = 2$, and $f(a) = f(0) = -(0)^3 + 3 = 3$; $f(b) = f(2) = -(2)^3 + 3 = -8 + 3 = 5$

The average rate of change is $\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} = \frac{5-3}{2} = 1$

Domains of functions

When a function f is given by a formula and its domain is not given, then it is assumed that the **domain is the largest set of real numbers for which f(x) can be computed and is a real number.**

To find a domain of a function (in this course) you should ask three questions

A) Does the formula contain a **variable in the denominator**?

If yes, the domain will consist of all real numbers except the ones that make the denominator zero.

To find the domain in this case:

- (i) Solve the equation: denominator = 0
- (ii) Write the domain: $Df = \{x | x \neq (\text{list the solutions found in (i)})\}$

Example: Find the domain of $f(x) = \frac{3x+2}{x^3-5x}$

- (i) $x^3 - 5x = 0$
 $x(x^2 - 5) = 0$
 $x = 0$ or $x^2 - 5 = 0$
 $x^2 = 5$
 $x = \pm \sqrt{5}$

- (ii) $Df = \{x | x \neq 0, \sqrt{5}, -\sqrt{5}\}$

B) Does the formula have a variable inside the **square root** (or even index radical)?

If yes, the domain will consist of all values x for which the radicand is nonnegative

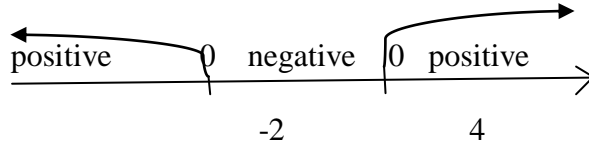
To find the domain in this case:

- (i) Solve the inequality: radicand ≥ 0
- (ii) Write the domain: $Df =$ solution from (i) in the interval notation

Example: Find the domain of $f(x) = \sqrt{x^2 - 2x - 8}$

- (i) $x^2 - 2x - 8 \geq 0$
 $x^2 - 2x - 8 = 0$
 $(x-4)(x+2) = 0$
 $x = 4$ $x = -2$

(ii)



(iii) $Df = (-\infty, -2] \cup [4, \infty)$

C) Does the formula contain a **logarithm**? We will deal with such functions in Chapter 6

Remarks: 1) If you **answer no** to questions A) and B) then the domain of the function is the set of all real numbers

2) If you answer yes to two of these questions, you must follow the procedure for each question and the domain of the function will be the *intersection* of all sets obtained in each question.

The **intersection of two sets** is the common part of those two sets, that is the set that contains numbers that belong to both sets.

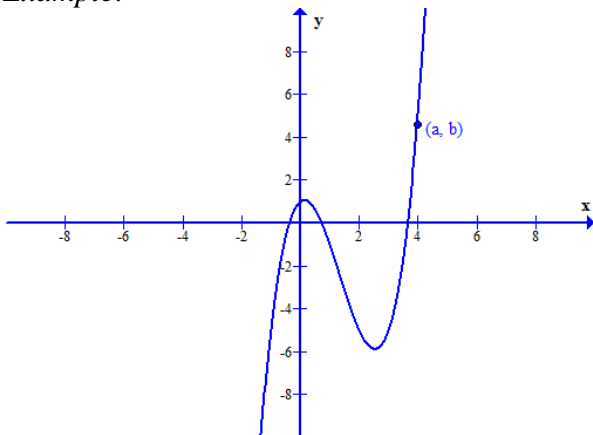
Example: The intersection of the red and green intervals, is the blue interval



3. 2 The graph of a function

The graph of a function f is the set of all points $(x, f(x))$, where x belongs to the domain of f

Example:

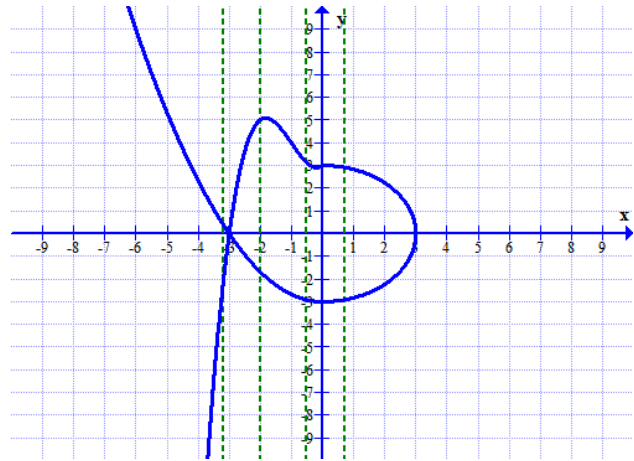


Since the point (a, b) is on the graph then $b = f(a)$

The graph of an equation is the graph of a function if it passes the **Vertical Line Test**. That is if every vertical line crosses the graph at most at one point.



A function



not a function

The **domain** of a function given by a graph is the set of all x , such that (x, y) , for some y , is on the graph.

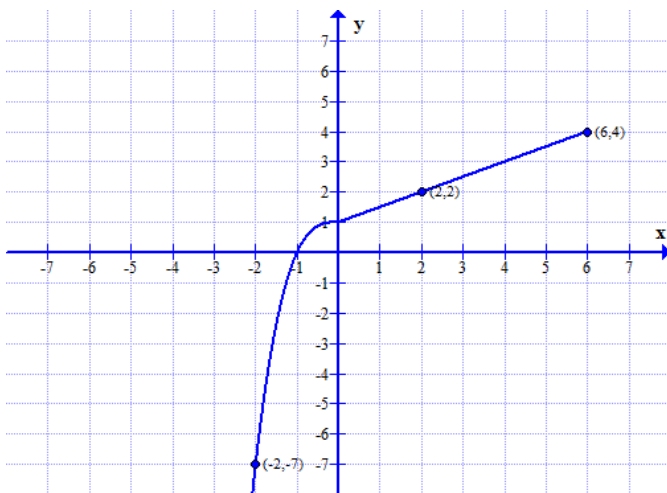
The **range** of a function is the set of all y , such that the point (x, y) , for some x , is on the graph.

To find value $f(a)$ from the graph, locate the point on the graph whose x -coordinate is a . The y -coordinate of that point is $f(a)$.

The **x -intercepts** are the points where the graph crosses or touches the x -axis. They are found by solving the equation $f(x) = 0$. They are the points at which f has value zero, and therefore they are often called the **zeros of function f** .

To find y -intercept, compute $f(0)$. A function can have at most one y -intercept.

Example: The graph of a function f is given below. Find the domain and the range. Find $f(-2)$. Find x - and y -intercepts, if any.



If you follow the graph from left to right, the x -coordinate changes from $-\infty$ to 6, so $D_f = (-\infty, 6]$. At the same time, the y -coordinate changes from $-\infty$ to 4, so Range of $f = (-\infty, 4]$.

To find $f(-2)$, find the point on the graph with x -coordinate -2 . This point is $(-2, -7)$. The y -coordinate of this point is -7 , hence $f(-2) = -7$.

The graph crosses x -axis at $(-1, 0)$, so the x -intercept is $(-1, 0)$. The y -intercept is $(0, 1)$

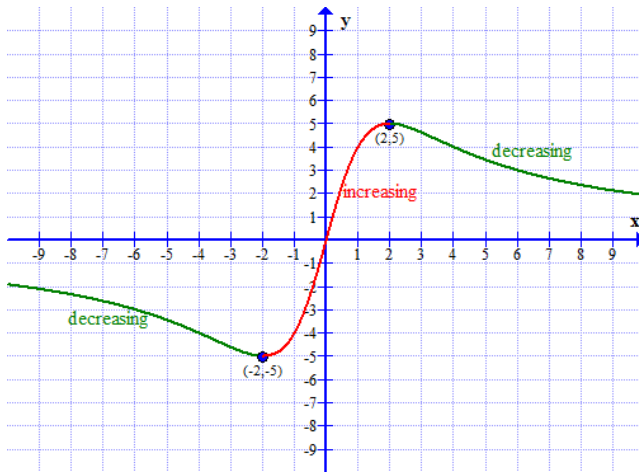
3.3 Properties of functions

A function is **increasing** on an interval (a, b) , if for every two values x_1 and x_2 in that interval, $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$ (y increases when x increases). A function is increasing on an interval if the graph is rising on this interval.

A function is **decreasing** on an interval (a, b) , if for every two values x_1 and x_2 in that interval, $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$ (y decreases when x increases). A function is decreasing on an interval, if the graph is falling on this interval.

A function is **constant** on an interval (a, b) , if for every two values x_1 and x_2 in that interval, $f(x_1) = f(x_2)$ (y remains the same). A function is constant on an interval, if the graph is horizontal on this interval.

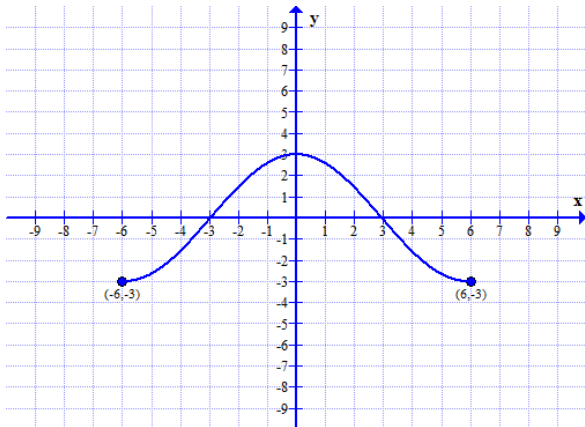
Example: Find the intervals on which the following function is increasing/ decreasing/constant



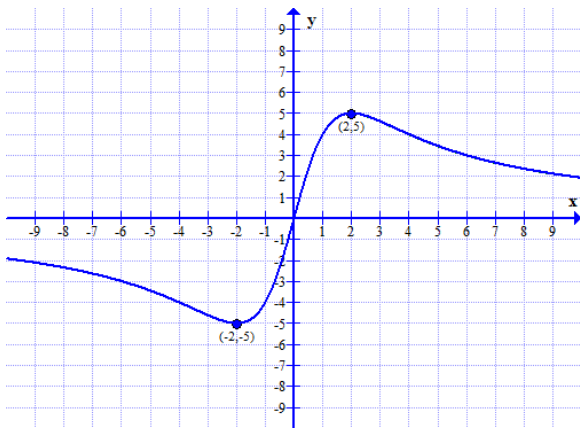
- (i) Identify the parts that are increasing/ decreasing/ constant
- (ii) Determine how the x-coordinate changes when you follow each part, and write it in the interval notation.

In the example above, graph is rising (function is increasing) when x- changes from -2 to 2, so f is increasing on $(-2, 2)$

A function f is **even**, if for every x in its domain, $-x$ is also in the domain and $f(-x) = f(x)$. If f is even, then its graph is **symmetric with respect to the y-axis**. The graph below shows an even function.



A function f is **odd**, if for every x in its domain, $-x$ is also in its domain and $f(-x) = -f(x)$. If f is an odd function, its graph is **symmetric with respect to the origin**. The graph below shows an odd function



To check **whether a function is odd/even/neither** follow the following steps:

- (i) Evaluate $f(-x)$ and simplify
- (ii) Compare $f(-x)$ with $f(x)$. If the formulas are the same then f is even. If the formulas are not the same, then f is not even.
- (iii) Write the formula for $-f(x)$
- (iv) Compare $f(-x)$ with $-f(x)$. If the formulas are the same then f is odd. If the formulas are not the same then f is not odd.

Remarks: i) If f is even, then it cannot be odd and vice versa

ii) There are functions that are neither odd nor even

Example: Check whether $f(x) = \frac{3x}{x^2 - 1}$ is even, odd or neither

(i) $f(-x) = \frac{3(-x)}{(-x)^2 - 1} = \frac{-3x}{x^2 - 1}$

(ii) $f(x)$ and $f(-x)$ are not the same, so f is not even

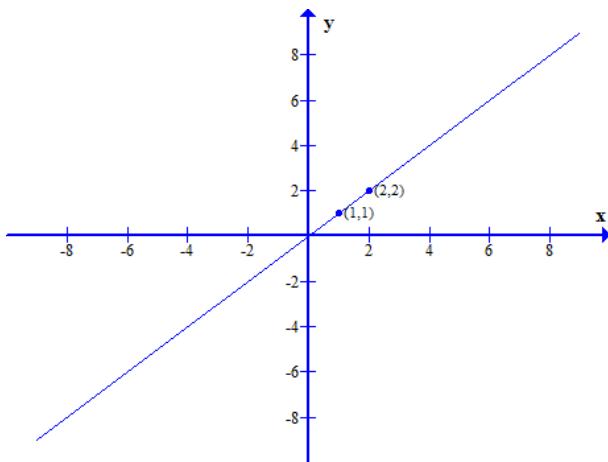
(iii) $-f(x) = -\frac{3x}{x^2 - 1} = \frac{-3x}{x^2 - 1}$

(iv) $f(-x)$ and $-f(x)$ are the same, so f is odd. Hence, the graph of $f(x)$ is symmetric with respect to the origin.

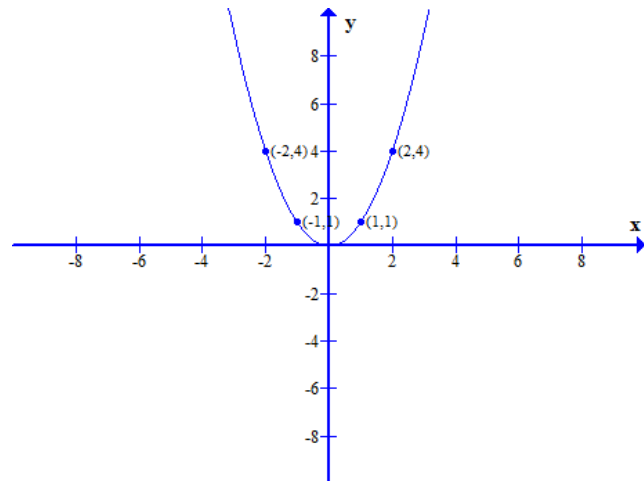
3.4 Library of functions; Piecewise defined functions

Here are the graphs that you must be able to recognize and draw without making the table.

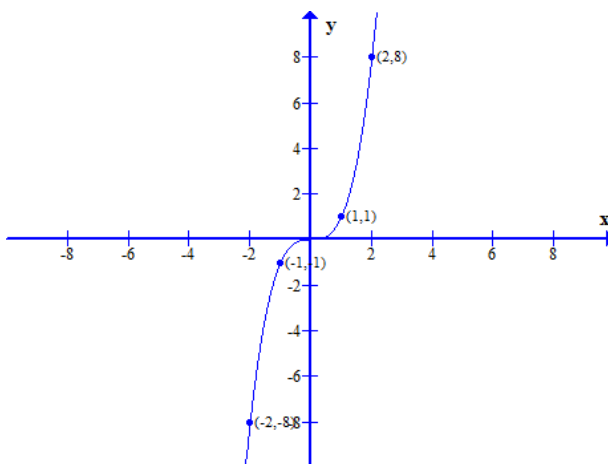
$f(x) = x$ identity function



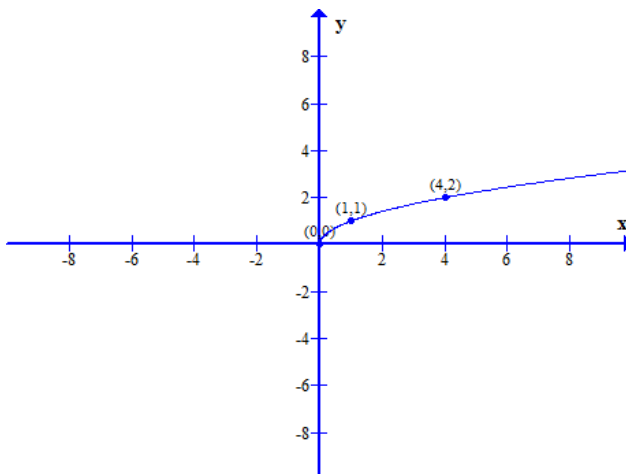
$f(x) = x^2$ square function



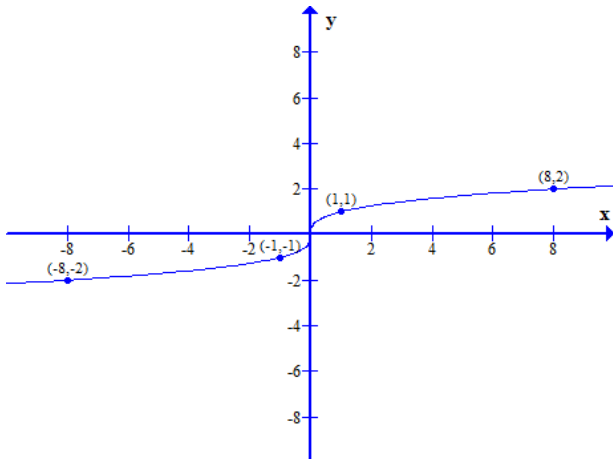
$f(x) = x^3$ cube function



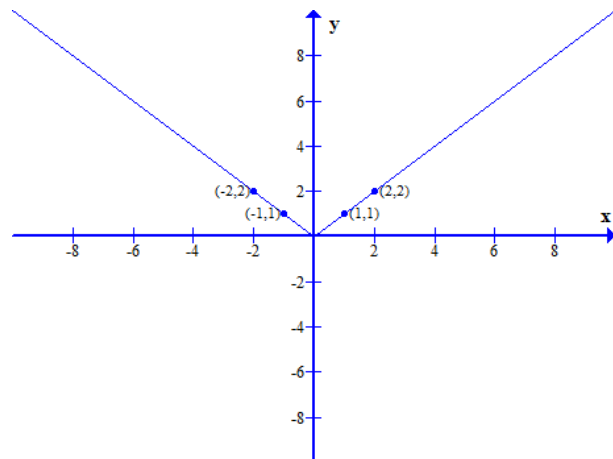
$f(x) = \sqrt{x}$ square root function



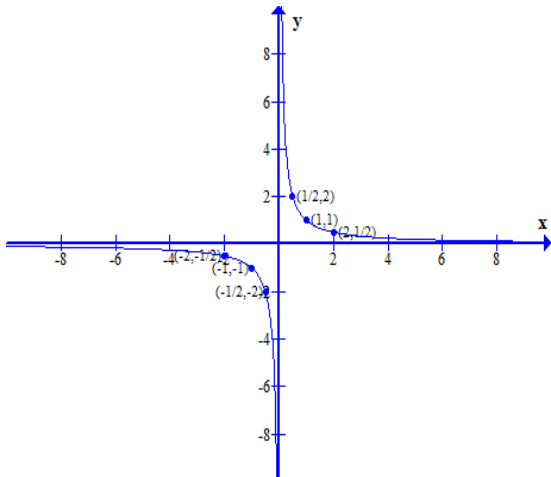
$$f(x) = \sqrt[3]{x} \text{ cube root function}$$



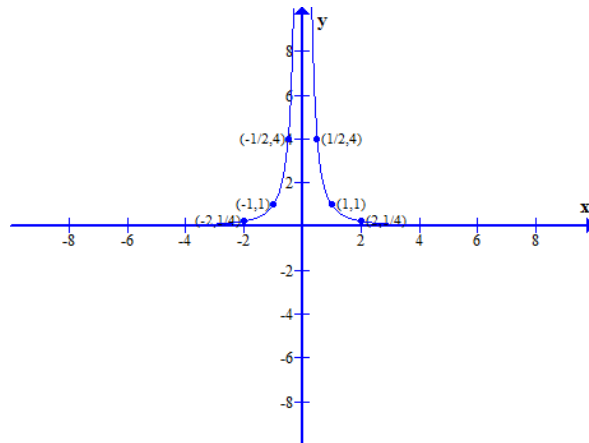
$$f(x) = |x| \text{ absolute value function}$$



$$f(x) = \frac{1}{x} \text{ reciprocal function}$$



$$f(x) = \frac{1}{x^2}$$



Piecewise functions

A function that is defined by more than one formula is called a piecewise function. Such functions are written using the brace {

Example:

$$f(x) = \begin{cases} 2x+3 & \text{if } x < 1 \\ |x| & \text{if } x \geq 1 \end{cases}$$

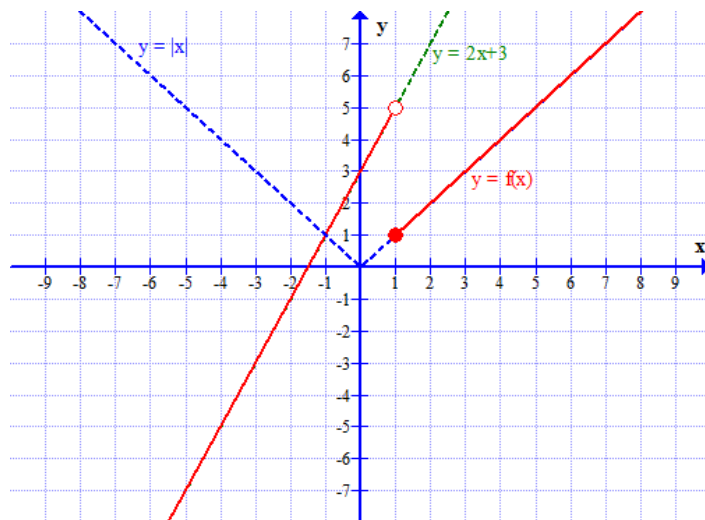
This function is defined by formula $f(x) = 2x+3$ only when $x < 1$, and by formula $f(x) = |x|$ only when $x \geq 1$.

To find $f(a)$, you must first determine whether $a < 1$ or $a \geq 1$. If $a < 1$, then you will use the first formula, $f(x) = 2x + 3$. If $a \geq 1$, then you will use the second formula, $f(x) = |x|$. For example, to evaluate $f(-3)$, we first notice that $-3 < 1$, so we use the formula reserved for $x < 1$, which is $f(x) = 2x + 3$.

Hence, $f(-3) = 2(-3) + 3 = -3$.

To graph a piecewise function: graph each function involved in the formula and take the piece of that graph that corresponds to the given values of x .

Example: The graph of $f(x)$ from previous example is in red in the picture below. The dashed graphs of $y = 2x + 3$ and $y = |x|$ should be erased.



3.5 Graphing techniques

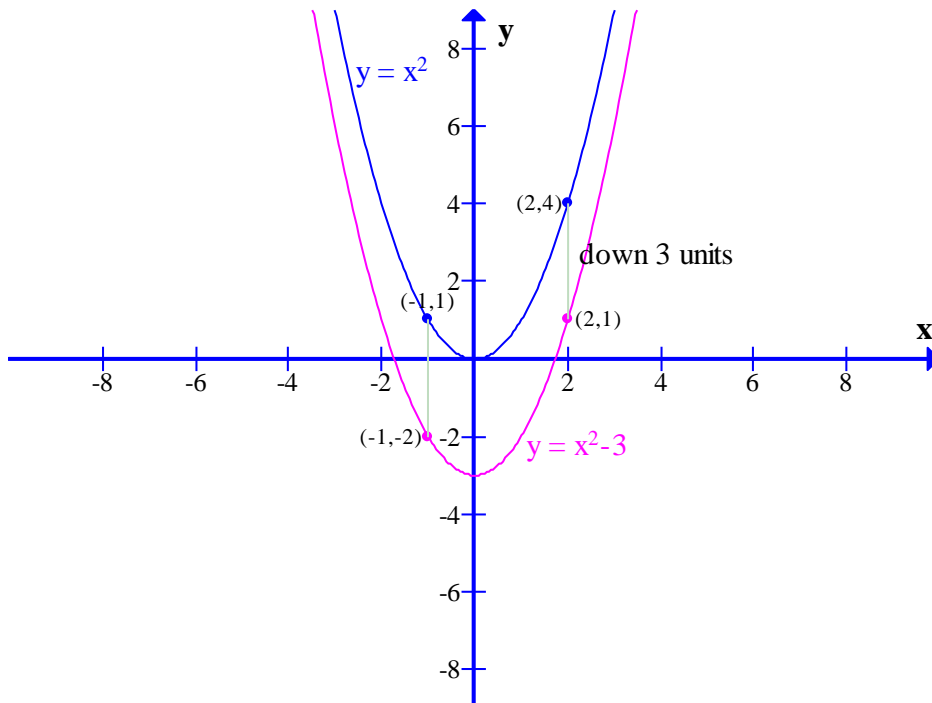
Graphs of certain functions can be obtained by graphing one of the functions in the library and performing some transformations on it.

Suppose c is a positive real number, $c > 0$. If the graph of a function f is given then to obtain the graph of

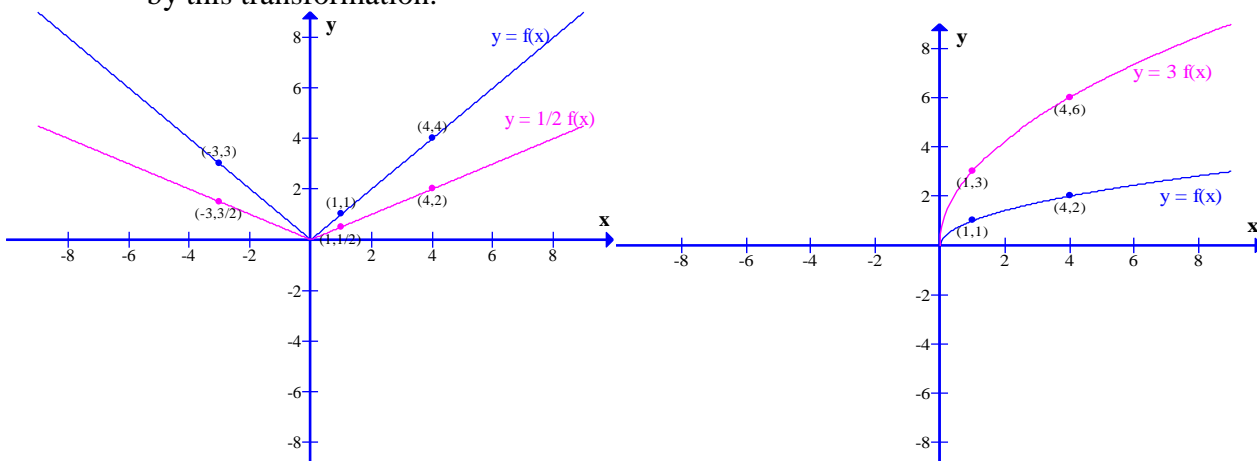
- (i) $g(x) = f(x) + c$ ($g(x) = f(x) - c$)
 shift the graph of f up (down) by c units.

The graph of g will have the same shape as that of f but it will be in a different place.

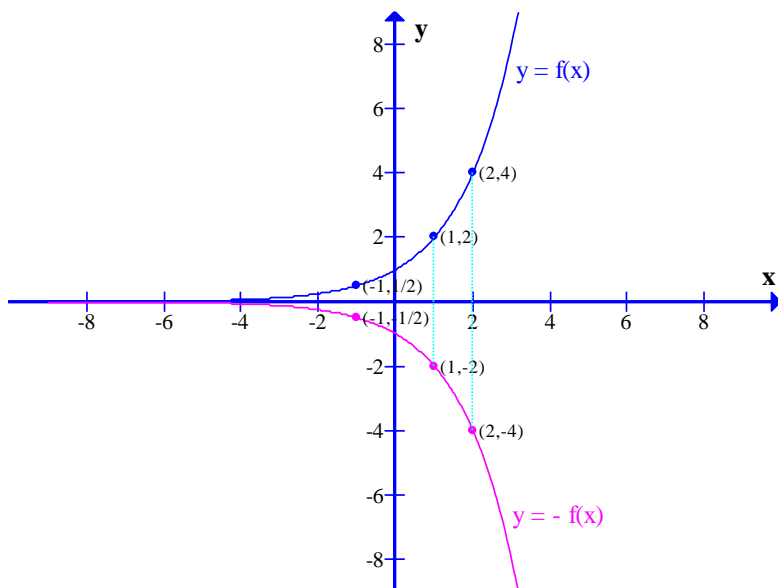
Each point on the graph of f will be moved vertically up (down). If (x,y) is on the graph of $f(x)$ then $(x, y + c)$ ($(x, y - c)$) will be on the graph of g .



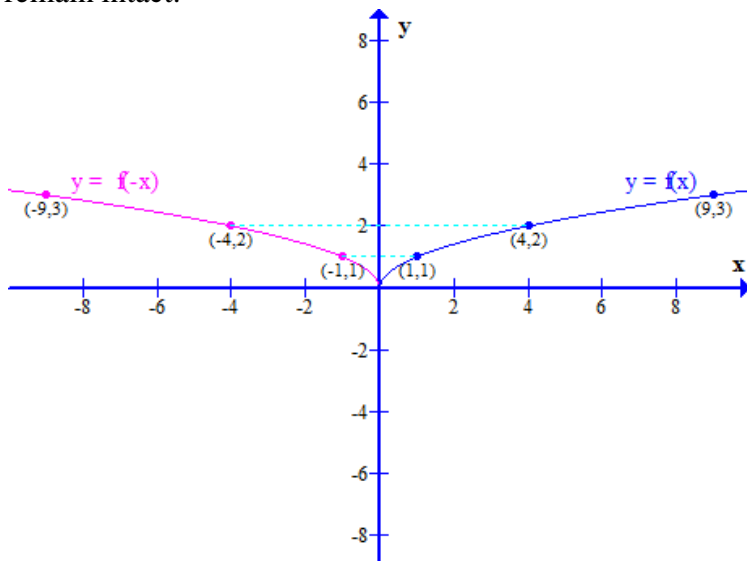
- (ii) $g(x) = c f(x)$
 stretch the graph of f vertically, if $c > 1$ and compress vertically when $0 < c < 1$. If the point (x,y) is on the graph of f then $(x, c y)$ is on the graph of g . The x -intercepts are not affected by this transformation.



- (iii) $g(x) = -f(x)$
 reflect the graph of f about the x -axis. If a point (x,y) is on the graph of f then $(x,-y)$ is on the graph of g . The graphs of f and g are symmetric with respect to the x -axis. The x -intercepts will remain intact.

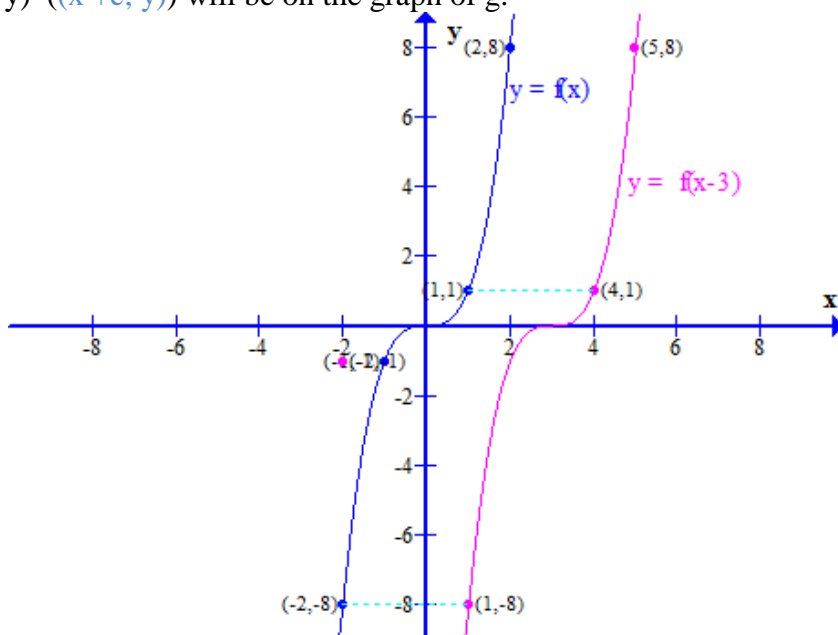


- (iv) $g(x) = f(-x)$
 reflect the graph of f about the y -axis. If a point (x,y) is on the graph of f then $(-x, y)$ is on the graph of g . The graphs of f and g are symmetric about the y -axis. The y -intercepts remain intact.



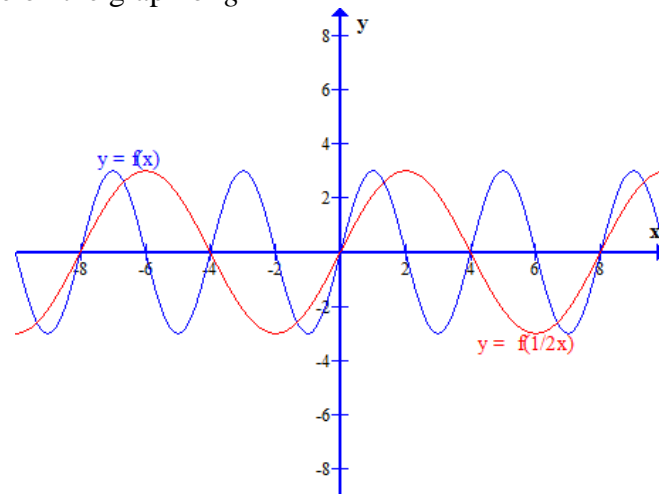
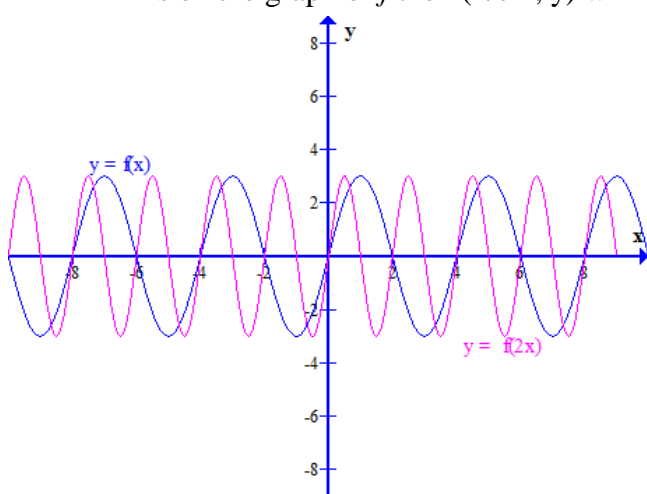
(v) $g(x) = f(x + c)$ ($g(x) = f(x - c)$)

Shift the graph of f horizontally to the left (right) by c units. Each point on the graph of f will be moved horizontally c units to the left (right). If (x, y) is on the graph of f then $(x - c, y)$ ($(x + c, y)$) will be on the graph of g .



(vi) $g(x) = f(cx)$

compress the graph of f horizontally, if $c > 1$ and stretch it horizontally if $0 < c < 1$. If (x, y) is on the graph of f then $(1/c x, y)$ will be on the graph of g



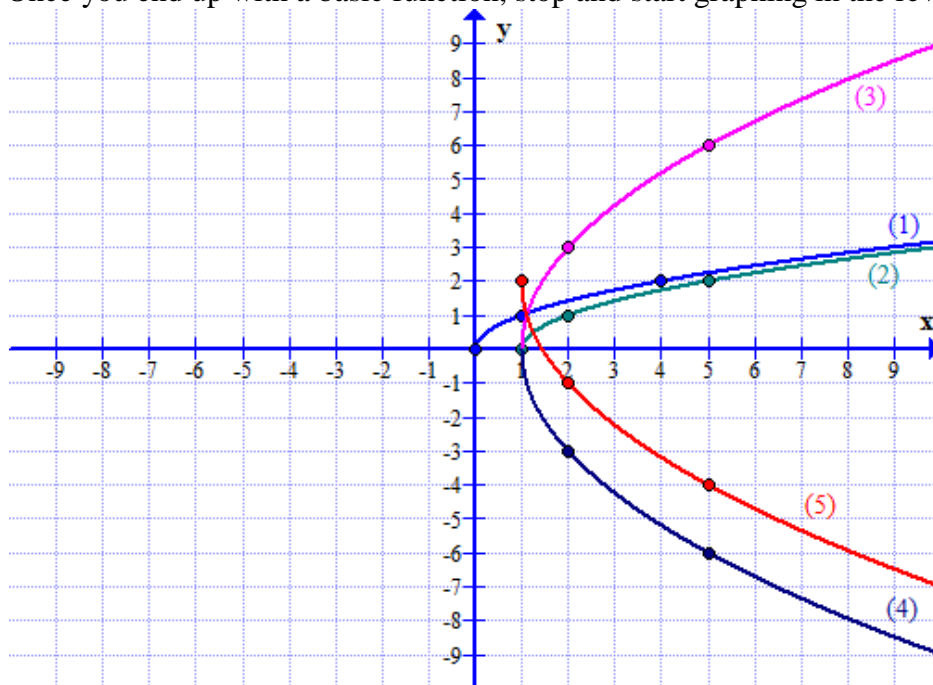
The transformations can be combined and performed in a sequence. If present, shift up/ down MUST be performed last.

Example: Use transformations to graph $f(x) = -3\sqrt{x-1} + 2$

You must determine first the order of transformations

- 1) (5) $f(x) = -3\sqrt{x-1} + 2$ A constant (2) is added to $y = -3\sqrt{x-1}$. This means that we need to graph $y = -3\sqrt{x-1}$ and move it 2 units up
- 2) (4) $y = -3\sqrt{x-1}$ Putting a minus sign in front of a function results in reflection about x-axis
We need to graph to $y = 3\sqrt{x-1}$ and reflect it about x-axis
- 3) (3) $y = 3\sqrt{x-1}$ $\sqrt{x-1}$ is multiplied by 3, so we have a vertical stretch. It is enough to graph $y = \sqrt{x-1}$ and stretch it by a factor of 3
- 4) (2) $y = \sqrt{x-1}$ replacing $x-1$ with x , gives us $y = \sqrt{x}$, which is a basic function. To get the graph of $y = \sqrt{x-1}$ we must shift the graph of $y = \sqrt{x}$ one unit to the right
- 5) (1) $y = \sqrt{x}$

Once you end up with a basic function, stop and start graphing in the reverse order: 5), 4), 3), 2) 1)

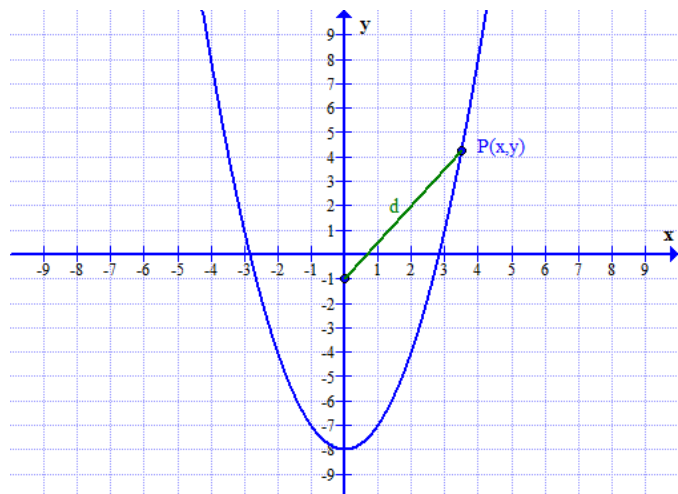


3.6 Mathematical models; building functions

In this section you will construct functions using known facts and the information given in a problem. The main direction will be “express quantity Q as a function of x ”. Your task will be to find a formula for Q in terms of x variable only, that is to write $Q = (\text{expression that contains } x \text{ only})$

Example: Let $P = (x,y)$ be a point on the graph of $y = x^2 - 8$. Express the distance d from P to the point $(0,-1)$ as a function of x . What is d , if $x = 0$.

As with any word problem, start with drawing a picture, if possible.



We must find the distance d . It is the distance between two points $(0,-1)$ and (x,y) . Using distance formula (sec 2.1) we get

$$d = \sqrt{(x-0)^2 + (y+1)^2}$$

$$d = \sqrt{x^2 + y^2 + 2y + 1}$$

Since d is to be a function of x , we need to eliminate the variable y . We need to find a relationship between x and y . We use the fact that since point (x,y) is on the graph of $y = x^2 - 8$, we must have $y = x^2 - 8$. Replacing y with $x^2 - 8$ in the formula for d , gives

$$d = \sqrt{x^2 + y^2 + 2y + 1} = \sqrt{x^2 + (x^2 - 8)^2 + 2(x^2 - 8) + 1}$$

$$d = \sqrt{x^2 + x^4 - 16x^2 + 64 + 2x^2 - 16 + 1}$$

$$d = \sqrt{x^4 - 13x^2 + 49}$$

The last formula expresses d in terms of x only, so d is expressed as a function of x .

If $x = 0$, then $d = \sqrt{(0)^4 - 13(0)^2 + 49} = \sqrt{49} = 7$