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Blow-up theorems for a structural acoustics model

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ABSTRACT

This article studies the finite time blow-up of weak solutions to a structural acoustics model consisting of a semilinear wave equation defined on a bounded domain $\Omega \subset \mathbb{R}^3$ which is strongly coupled with a Berger plate equation acting on the elastic wall, namely, a flat portion of the boundary. The system is influenced by several competing forces, including boundary and interior source and damping terms. We stress that the power-type source term acting on the wave equation is allowed to have a *supercritical* exponent, in the sense that its associated Nemytskii operator is not locally Lipschitz from H^1 into L^2. In this paper, we prove the blow-up results for weak solutions when the source terms are stronger than damping terms, by considering two scenarios of the initial data: (i) the initial total energy is negative; (ii) the initial total energy is positive but small, while the initial quadratic energy is sufficiently large. The most significant challenge in this work arises from the coupling of the wave and plate equations on the elastic wall.

1. Introduction

We study the finite time blow-up of weak solutions for a structural acoustics model influenced with nonlinear forces. Precisely, we consider the following coupled system of nonlinear PDEs:

$$\begin{cases} u_{tt} - \Delta u + u + g_1(u_t) = f(u) & \text{in } \Omega \times (0,T), \\ w_{tt} + \Delta^2 w + g_2(w_t) + u_t|_\Gamma = h(w) & \text{in } \Gamma \times (0,T), \\ u = 0 & \text{on } \Gamma_0 \times (0,T), \\ \partial_\nu u = w_t & \text{on } \Gamma \times (0,T), \\ w = \partial_{\nu_\Gamma} w = 0 & \text{on } \partial\Gamma \times (0,T), \\ (u(0), u_t(0)) = (u_0, u_1), \qquad (w(0), w_t(0)) = (w_0, w_1), \end{cases} \tag{1.1}$$

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where the initial data reside in the finite energy space, i.e.,

$$(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \text{ and } (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma).$$

The space $H^1_{\Gamma_0}(\Omega)$ defined in (2.1) consists of all H^1 functions that vanish on Γ_0.

Here, $\Omega \subset \mathbb{R}^3$ is a bounded, open, connected domain with smooth boundary $\partial\Omega = \overline{\Gamma_0 \cup \Gamma}$, where Γ_0 and Γ are two disjoint, open, connected sets of positive Lebesgue measure. Moreover, Γ is a *flat* portion of the boundary of Ω and is referred to as the elastic wall. The part Γ_0 of the boundary $\partial\Omega$ describes a rigid wall, while the coupling takes place on the flexible wall Γ.

The nonlinearities f and h are source terms acting on the wave and plate equations respectively, where the source term $f(u)$ is of a supercritical order, in the sense that its associated Nemytskii operator is not locally Lipschitz from $H^1_{\Gamma_0}(\Omega)$ into $L^2(\Omega)$. In the case of the 3D domain Ω, the supercritical order means that the exponent of the power-like function f is larger than 3. We stress that both source terms $f(u)$ and $h(w)$ are allowed to have "bad" signs which may cause instability (blow up) in finite time. In addition, the system is influenced by two other forces, namely $g_1(u_t)$ and $g_2(w_t)$ representing frictional damping terms acting on the wave and plate equations, respectively. The vectors ν and ν_Γ denote the outer normals to Γ and $\partial\Gamma$, respectively.

Models such as (1.1) arise in the context of modeling gas pressure in an acoustic chamber which is surrounded by a combination of rigid and flexible walls. The pressure in the chamber is described by the solution to a wave equation, while vibrations of the flexible wall are described by the solution to a coupled Berger plate equation.

Differential equations describing structural acoustic interaction have rich history. These models are well known in both the physical and mathematical literature and go back to the canonical models considered in [6,7,22]. In particular, in a pioneering work [6], Banks et al. introduced a 2D model to describe acoustic-structure interaction in an acoustic cavity of rectangular shape where the boundary consists of hard walls on three sides, and a vibrating wall on the fourth side which is modeled by an Euler-Bernoulli beam equation. Piezoceramic patches are attached to the beam to control structural vibrations and the acoustic pressure in the model. In the context of stabilization and controllability of structural acoustics models there is a very large body of literature. We refer the reader to the monograph by Lasiecka [26] which provides a comprehensive overview and quotes many works on these topics. Other related contributions include [2–5,12, 25]. However, to the best of our knowledge, the finite time blow-up for structural acoustics models under the influence of nonlinear forces has not been studied in the literature, and so we address this issue in this paper.

Our goal is to understand the source-damping interactions in the structural acoustics model (1.1), and how these interactions affect the behaviors of weak solutions. The local well-posedness of weak solutions to system (1.1) was proved by Becklin and Rammaha in [8], in which they also showed the global existence if the damping are more dominant than the source terms. In our paper [13], by using the potential well theory, we proved the global existence of weak solutions and estimated the energy decay rates, provided the initial data come from the stable part of the potential well. In the present manuscript, we shall demonstrate the blow-up phenomena of weak solutions when the source terms are stronger than damping terms, and we consider two cases of the initial data: (i) the initial total energy is negative, which means that the initial potential energy due to the nonlinear forces is sufficiently large; (ii) the initial total energy is positive but small enough, while the initial quadratic energy is large. In this case, the initial data come from the unstable part of the potential well. To understand these blow-up phenomena intuitively, one can imagine that a nonlinear force continuously acts on the elastic wall to increase its vibration and simultaneously another nonlinear force acts on the gas inside the acoustic chamber to increase its pressure, and these forces surpass the damping effects, then the system collapses at some finite time.

The difficulty for proving the blow-up of weak solutions to system (1.1) comes from the coupling of the wave equation and the plate equation on the elastic wall, i.e., the flat portion of the boundary. We notice that the coupling of these two evolution equations in (1.1) are through the term $u_t|_\Gamma$ where Γ is the elastic wall. Since we consider weak solutions of the wave equation, u_t belongs to $L^2(\Omega)$, while a generic L^2 function may not have a well-defined trace on the boundary of Ω. In system (1.1), the term $u_t|_\Gamma$ is defined in a weak sense via the plate equation. But we do not have an appropriate estimate for the $L^2(\Gamma)$ norm of $u_t|_\Gamma$. Therefore, throughout the proof of our blow-up results, we strive to prevent directly estimating $u_t|_\Gamma$, and the idea is to convert $u_t|_\Gamma$ to a different term by taking advantage of the structure of the equation. Our basic strategy for proving the blow-up is to show the function $Y(t) = G^{1-a}(t) + \varepsilon N'(t)$ defined in (3.9) approaches infinity in finite time, by deducing a differential inequality of the form $Y'(t) \geq Y^\mu(t)$ with $\mu > 1$. Such an "anti-Lyapunov" function $Y(t)$ was originally constructed in [16] by Georgiev and Todorova, who paved the way to the analysis of interactions of nonlinear damping and source terms in wave equations. Here, $G(t) = -\mathcal{E}(t)$ where $\mathcal{E}(t)$ is the total energy. But, in [16] and many other related works in the literature, $N(t)$ is usually defined as the L^2 norm of the unknown function. However, in our argument, we use a trick by including an additional term $\int_0^t \int_\Gamma \gamma u(\tau) \cdot w(\tau) d\Gamma d\tau$ in $N(t)$ (see (3.2)). This extra term helps us to convert the troublesome term $u_t|_\Gamma$ in our estimate to a well-behaved term w_t, where the $L^2(\Gamma)$ norm of w_t is part of the energy.

The reader may also refer to works by Levine and Serrin [29], and by Vitillaro [30], which provide some classical results on global nonexistence or finite-time blow-up for nonlinear hyperbolic problems *with dissipation*. Moreover, Glassey [14,15], Levine [28] and Keller [23] have classical results on blow-up of nonlinear wave equations *with no damping*, using various arguments.

We must point out that in the original linear structural acoustics model, u satisfies the wave equation $u_{tt} - \Delta u = 0$. However, in order to resolve some technical difficulty occurred during our proof for the blow-up results, we add a term u to the linear part, and the linear operator in our system (1.1) becomes $u_{tt} - \Delta u + u$, which usually appears in a Klein-Gordon equation. The extra term u in the linear operator is useful when we estimate the $L^2(\Gamma)$ norm of the trace of u in (3.19) since it allows us to obtain precise coefficients on the right-hand side of the inequality, which is critical for our argument.

Source-damping interactions have important applications. On one hand, in control theory, one may use damping terms to stabilize the system. On the other hand, one can create instability by strengthening the source terms. The interesting source-damping interactions in wave equations have been illustrated by a pioneering work by Georgiev and Todorova [16]. Bociu and Lasiecka wrote a series of papers [9–11] to study wave equations with *supercritical* source and damping terms acting in the interior and on the boundary of the domain. Also, Guo [17] proved the global well-posedness of a 3D wave equation with a source term of an arbitrarily large exponent, as long as the frictional damping term is strong enough to suppress the growth of solutions due to the source term. One may also refer to papers [18–20] for source-damping interactions in coupled wave equations. In [31–33], Vitillaro studied the wave equation with hyperbolic dynamical boundary conditions, and showed local and global well-posedness and blow-up results, depending on the choice of different growth rates of interior and boundary nonlinear damping and sources.

Let us mention an interesting work [27] by Lasiecka and Rodrigues about a structural-acoustic wall problem in 3D, in which the structural wall is modeled by a 2D Kirchhoff-Boussinesq plate. The paper [27] addresses the competition between a Boussinesq forcing term $\Delta\{w^2\}$ (which may lead to finite-time blow-up) and a restoring internal force $\text{div}\{|\nabla w|^2 \nabla w\}$ (which has stabilizing effect for low frequencies) in the plate equation defined on the wall, and such a competition between these two forces determines the global behavior of the model.

The content of the paper is organized as follows. In Section 2, we state well-posedness results from [8] by Becklin and Rammaha, and our previous results on global existence and energy decay of weak solutions in [13]. Moreover, we state main results of this manuscript, namely, finite-time blow-up of weak solutions. In Section 3, we prove the blow-up of weak solutions by assuming the initial total energy is negative. In Section 4, we show the finite-time blow-up by supposing the initial total energy is positive.

2. Preliminaries and main results

2.1. Notation

Throughout the paper the following notational conventions for L^p space norms and standard inner products will be used:

$$||u||_p = ||u||_{L^p(\Omega)}, \qquad (u,v)_\Omega = (u,v)_{L^2(\Omega)},$$
$$|u|_p = ||u||_{L^p(\Gamma)}, \qquad (u,v)_\Gamma = (u,v)_{L^2(\Gamma)}.$$

We also use the notation γu to denote the *trace* of u on Γ. As is customary, C always denotes a generic positive constant which may change from line to line.

Further, we put

$$H^1_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}, \tag{2.1}$$

and $\|u\|_{H^1_{\Gamma_0}(\Omega)} := (\|\nabla u\|_2^2 + \|u\|_2^2)^{\frac{1}{2}}$. It is well-known that the standard norm $\|u\|_{H^1_{\Gamma_0}(\Omega)}$ is equivalent to $\|\nabla u\|_2$. For a similar reason, we put $\|w\|_{H^2_0(\Gamma)} = |\Delta w|_2$.

In the proof, the following Sobolev imbeddings will be used: $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^6(\Omega)$ and $H^1(\Gamma) \hookrightarrow L^q(\Gamma)$ for any $1 \le q < \infty$.

2.2. Well-posedness of weak solutions

Throughout the paper, we study (1.1) under the following assumptions.

Assumption 2.1.

Damping: *g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ are continuous and monotone increasing functions with $g_1(0) = g_2(0) = 0$. In addition, the following growth conditions at infinity hold: there exist positive constants α and β such that, for $|s| \ge 1$,*

$$\alpha|s|^{m+1} \le g_1(s)s \le \beta|s|^{m+1}, \text{ with } m \ge 1,$$
$$\alpha|s|^{r+1} \le g_2(s)s \le \beta|s|^{r+1}, \text{ with } r \ge 1.$$

Source terms: *f and h are functions in $C^1(\mathbb{R})$ such that*

$$|f'(s)| \le C(|s|^{p-1} + 1), \text{ with } 1 \le p < 6,$$
$$|h'(s)| \le C(|s|^{q-1} + 1), \text{ with } 1 \le q < \infty.$$

Parameters: $p\frac{m+1}{m} < 6$.

The following assumption will be needed for establishing an uniqueness result.

Assumption 2.2. *For $p > 3$, we assume that $f \in C^2(\mathbb{R})$ with $|f''(u)| \le C(|u|^{p-2} + 1)$ for all $u \in \mathbb{R}$.*

Remark 2.3. The assumption that $p\frac{m+1}{m} < 6$ is needed for the local existence of weak solutions in the finite energy space, when considering the interior source term of a *supercritical* order, namely, $p \in [3, 6)$. Here, the upper bound of the range of p is due to the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ in 3D. Notice that, if the exponent p of the source term is close to 6, then the exponent m of the damping term must be very large to satisfy the assumption, in order to guarantee that the system is locally solvable. The original works that have developed the techniques of handling this scenario for wave equations are in [9–11]. On the other hand, regarding the exponent q of the boundary source term acting on the plate equation, q is allowed to be any large number, because in 2D, $H^1(\Gamma)$ is imbedded in $L^s(\Gamma)$ for any $s \geq 2$.

We begin by introducing the definition of a suitable weak solution for (1.1).

Definition 2.4. A pair of functions (u, w) is said to be a weak solution of (1.1) on the interval $[0, T]$ provided:

(i) $u \in C([0, T]; H^1_{\Gamma_0}(\Omega))$, $u_t \in C([0, T]; L^2(\Omega)) \cap L^{m+1}(\Omega \times (0, T))$,
(ii) $w \in C([0, T]; H^2_0(\Gamma))$, $w_t \in C([0, T]; L^2(\Gamma)) \cap L^{r+1}(\Gamma \times (0, T))$,
(iii) $(u(0), u_t(0)) = (u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$,
(iv) $(w(0), w_t(0)) = (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma)$,
(v) The functions u and w satisfy the following variational identities for all $t \in [0, T]$:

$$
\begin{aligned}
&(u_t(t), \phi(t))_\Omega - (u_1, \phi(0))_\Omega - \int_0^t (u_t(\tau), \phi_t(\tau))_\Omega d\tau + \int_0^t (\nabla u(\tau), \nabla\phi(\tau))_\Omega d\tau \\
&\quad + \int_0^t (u(\tau), \phi(\tau))_\Omega d\tau - \int_0^t (w_t(\tau), \gamma\phi(\tau))_\Gamma d\tau + \int_0^t \int_\Omega g_1(u_t(\tau))\phi(\tau) dx d\tau \\
&\quad = \int_0^t \int_\Omega f(u(\tau))\phi(\tau) dx d\tau, \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
&(w_t(t) + \gamma u(t), \psi(t))_\Gamma - (w_1 + \gamma u_0, \psi(0))_\Gamma - \int_0^t (w_t(\tau), \psi_t(\tau))_\Gamma d\tau \\
&\quad - \int_0^t (\gamma u(\tau), \psi_t(\tau))_\Gamma d\tau + \int_0^t (\Delta w(\tau), \Delta\psi(\tau))_\Gamma d\tau \\
&\quad + \int_0^t \int_\Gamma g_2(w_t(\tau))\psi(\tau) d\Gamma d\tau = \int_0^t \int_\Gamma h(w(\tau))\psi(\tau) d\Gamma d\tau, \qquad (2.3)
\end{aligned}
$$

for all test functions ϕ and ψ satisfying: $\phi \in C([0, T]; H^1_{\Gamma_0}(\Omega)) \cap L^{m+1}(\Omega \times (0, T))$, $\psi \in C\left([0, T]; H^2_0(\Gamma)\right)$ with $\phi_t \in L^1(0, T; L^2(\Omega))$, and $\psi_t \in L^1(0, T; L^2(\Gamma))$.

Our work in this paper is based on the existence results which were established in [8] by Becklin and Rammaha. For the reader's convenience, we first summarize the important results in [8].

Theorem 2.5 *(Local and global weak solutions [8]). Under the validity of Assumption 2.1, then there exists a local weak solution (u, w) to (1.1) defined on $[0, T_0]$ for some $T_0 > 0$ depending on the initial energy $E(0)$, where the quadratic energy $E(t)$ is given by*

$$
E(t) := \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|u(t)\|_2^2 + |w_t(t)|_2^2 + |\Delta w(t)|_2^2\right). \qquad (2.4)
$$

- (u, w) *satisfies the following energy identity for all* $t \in [0, T_0]$*:*

$$E(t) + \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau$$
$$= E(0) + \int_0^t \int_\Omega f(u)u_t dx d\tau + \int_0^t \int_\Gamma h(w)w_t d\Gamma d\tau. \tag{2.5}$$

- *In addition to Assumption 2.1, if we assume that* $u_0 \in L^{p+1}(\Omega)$, $p \leq m$ *and* $q \leq r$*, then the said solution* (u, w) *is a global weak solution and* T_0 *can be taken arbitrarily large.*
- *If Assumptions 2.1 and 2.2 are valid, and if we further assume that* $u_0 \in L^{\frac{3(p-1)}{2}}(\Omega)$*, then weak solutions of (1.1) are unique.*
- *If Assumption 2.1 is valid, and if we additionally assume that* $u_0 \in L^{3(p-1)}(\Omega)$ *and* $m \geq 3p - 4$ *when* $p > 3$*, then weak solutions of (1.1) are unique.*

Remark 2.6. Energy equality (2.5) is critical for the proof of the blow-up results in this paper. The energy equality can be "formally" derived by multiplying the wave and plate equations by u_t and w_t respectively and integrating over the domains. However, since we consider weak solutions, it is not easy to justify such a formal derivation. The method of proving the energy equality in [8] is to approximate u_t and w_t by difference quotients. A main difficulty comes from the boundary condition $\partial_\nu u = w_t$ on Γ, and $u_t|_\Gamma$, the wave velocity trace on Γ. These troublesome terms have been delicately handled in [8]. Please refer to [8] for the proof of the energy equality. Some important elements of the difference quotient method can be found in [24] by Koch and Lasiecka.

2.3. Potential well solutions

In this subsection we briefly discuss the potential well theory which originates from the theory of elliptic equations. In order to do so, we need to impose additional assumptions on the source terms $f(u)$ and $h(w)$.

Assumption 2.7.

- *There exists a nonnegative function* $F(u) \in C^1(\mathbb{R})$ *such that* $F'(u) = f(u)$*, and* F *is homogeneous of order* $p + 1$*, i.e.,* $F(\lambda u) = \lambda^{p+1}F(u)$*, for* $\lambda > 0,\ u \in \mathbb{R}$*.*
- *There exists a nonnegative function* $H(s) \in C^1(\mathbb{R})$ *such that* $H'(s) = h(s)$*, and* H *is homogeneous of order* $q + 1$*, i.e.,* $H(\lambda s) = \lambda^{q+1}H(s)$*, for* $\lambda > 0,\ s \in \mathbb{R}$*.*

Remark 2.8. From Euler homogeneous function theorem we infer that

$$uf(u) = (p + 1)F(u), \quad wh(w) = (q + 1)H(w). \tag{2.6}$$

Because of Assumption 2.1 and the homogeneity of F and H, we obtain that there exists a positive constant M such that

$$F(u) \leq M|u|^{p+1}, \quad H(w) \leq M|w|^{q+1}. \tag{2.7}$$

Moreover, due to (2.6), f is homogeneous of order p and h is homogeneous of order q satisfying

$$|f(u)| \leq M(p + 1)|u|^p, \quad |h(w)| \leq M(q + 1)|w|^q. \tag{2.8}$$

Recall the quadratic energy $E(t)$ has been introduced in (2.4). Now, we define the total energy $\mathcal{E}(t)$ of system (1.1) by

$$\begin{aligned}\mathcal{E}(t) :&= E(t) - \int_\Omega F(u(t))dx - \int_\Gamma H(w(t))d\Gamma \\ &= \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|u(t)\|_2^2 + |w_t(t)|_2^2 + |\Delta w(t)|_2^2\right) \\ &\quad - \int_\Omega F(u(t))dx - \int_\Gamma H(w(t))d\Gamma. \end{aligned} \tag{2.9}$$

Then, the energy identity (2.5) is equivalent to

$$\mathcal{E}(t) + \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau = \mathcal{E}(0). \tag{2.10}$$

With $X := H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$, we define the functional $\mathcal{J} : X \to \mathbb{R}$ by

$$\mathcal{J}(u,w) := \frac{1}{2}(\|\nabla u(t)\|_2^2 + \|u\|_2^2 + |\Delta w(t)|_2^2) - \int_\Omega F(u(t))dx - \int_\Gamma H(w(t))d\Gamma, \tag{2.11}$$

where $\mathcal{J}(u,w)$ is the potential energy of the system. Then we have

$$\mathcal{E}(t) = \mathcal{J}(u,w) + \frac{1}{2}(\|u_t(t)\|_2^2 + |w_t(t)|_2^2). \tag{2.12}$$

The Fréchet derivative of $\mathcal{J}$ at $(u,w) \in X$ is given by

$$\begin{aligned}\langle \mathcal{J}'(u,w),(\phi,\psi)\rangle &= \int_\Omega \nabla u \cdot \nabla \phi dx + \int_\Gamma \Delta w \cdot \Delta \psi d\Gamma + \int_\Omega u\phi dx \\ &\quad - \int_\Omega f(u)\phi dx - \int_\Gamma h(w)\psi d\Gamma, \end{aligned} \tag{2.13}$$

for $(\phi,\psi) \in X$. The *Nehari manifold* $\mathcal{N}$ can be defined by

$$\mathcal{N} := \{(u,w) \in X\backslash\{(0,0)\} : \langle \mathcal{J}'(u,w),(u,w)\rangle = 0\},$$

which along with (2.13) gives

$$\begin{aligned}\mathcal{N} = \Big\{(u,w) \in X\backslash\{(0,0)\} : \|\nabla u\|_2^2 + \|u\|_2^2 + |\Delta w|_2^2 = (p+1)\int_\Omega F(u)dx \\ + (q+1)\int_\Gamma H(w)d\Gamma\Big\}. \end{aligned} \tag{2.14}$$

By Lemma 2.8 in our paper [13] and Lemma 2.7 in [19], the depth of the potential well d is positive and satisfies

$$d := \inf_{(u,w)\in\mathcal{N}} \mathcal{J}(u,w) = \inf_{(u,w)\in X\setminus\{(0,0)\}} \sup_{\lambda\geq 0} \mathcal{J}(\lambda(u,w)) > 0, \tag{2.15}$$

for $1 < p \leq 5,\ q > 1$.

We define

$$\mathcal{W} := \{(u,w) \in X : \mathcal{J}(u,w) < d\},$$

$$\mathcal{W}_1 := \left\{(u,w) \in \mathcal{W} : \|\nabla u\|_2^2 + \|u\|_2^2 + |\Delta w|_2^2 > (p+1)\int_\Omega F(u)dx + (q+1)\int_\Gamma H(w)d\Gamma\right\} \cup \{(0,0)\},$$

$$\mathcal{W}_2 := \left\{(u,w) \in \mathcal{W} : \|\nabla u\|_2^2 + \|u\|_2^2 + |\Delta w|_2^2 < (p+1)\int_\Omega F(u)dx + (q+1)\int_\Gamma H(w)d\Gamma\right\}.$$

It is obvious that $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{W}$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$. We call $\mathcal{W}$ the potential well and d is the depth of the well. We call $\mathcal{W}_1$ the stable part of the potential well, and $\mathcal{W}_2$ the unstable part of the potential well.

For initial data coming from the stable part of the potential well, we have proved the following result of global solutions in [13].

Theorem 2.9 *(Potential well solutions [13]). Assume that Assumption 2.1 and Assumption 2.7 hold. Let $1 < p \leq 5$ and $q > 1$. Assume further $(u_0, w_0) \in \mathcal{W}_1$ and $\mathcal{E}(0) < d$. Then system (1.1) admits a global solution (u, w). In addition, for any $t \geq 0$, we have*

$$\begin{cases} (i)\ \mathcal{J}(u,w) \leq \mathcal{E}(t) \leq \mathcal{E}(0), \\ (ii)\ (u,w) \in \mathcal{W}_1, \\ (iii)\ E(t) \leq \dfrac{cd}{c-2}, \\ (iv)\ \dfrac{c-2}{c}E(t) \leq \mathcal{E}(t) \leq E(t), \end{cases}$$

where $c = \min\{p+1, q+1\} > 2$.

In paper [13], we also studied the energy decay rates for potential well solutions.

It is shown in Theorem 2.9 the invariance of $\mathcal{W}_1$ under the dynamics. In fact we have the same result for $\mathcal{W}_2$.

Lemma 2.10. *Assume that Assumption 2.1 and Assumption 2.7 hold. Let $1 < p \leq 5$ and $q > 1$. Assume further $(u_0, w_0) \in \mathcal{W}_2$ and $\mathcal{E}(0) < d$. Then the weak solution $(u(t), w(t))$ is in $\mathcal{W}_2$ for all $t \in [0, T)$, where $[0, T)$ is the maximal interval of existence.*

Proof. Please see the Appendix. □

Remark 2.11. For initial values coming from the unstable part $\mathcal{W}_2$ of the potential well, we shall state a blow-up result, namely Corollary 2.17.

2.4. Main results

Our first result is the blow-up of solutions if the source terms are stronger than damping terms, and the initial energy is negative. In order to state our first blow-up result, we need additional assumptions on the source terms.

Assumption 2.12.

- *There exists a function $F(u) \in C^1(\mathbb{R})$ such that $F'(u) = f(u)$. In addition, there exist $c_0 > 0$ and $c_1 > 3$ such that*

$$F(u) \geq c_0|u|^{p+1}, \quad uf(u) \geq c_1 F(u), \quad \forall\ u \in \mathbb{R}. \tag{2.16}$$

- *There exists a function $H(s) \in C^1(\mathbb{R})$ such that $H'(s) = h(s)$. In addition, there exist $c_2 > 0$ and $c_3 > 3$ such that*

$$H(s) \geq c_2|s|^{q+1}, \quad sh(s) \geq c_3 H(s), \quad \forall\ s \in \mathbb{R}. \tag{2.17}$$

The following blow-up result shows that if the initial energy is negative, and the source terms are more dominant than their corresponding damping terms, then every weak solution of (1.1) blows up in finite time.

Theorem 2.13 *(Blow-up with negative initial energy). Suppose that Assumption 2.1 and Assumption 2.12 hold. Assume $p > m$, $q > r$, and $\mathcal{E}(0) < 0$. Then the weak solution $(u(t), w(t))$ of system (1.1) blows up in finite time. In particular,*

$$\limsup_{t \to T^-} (\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2) = +\infty,$$

for some $0 < T < \infty$.

Remark 2.14. Combining the requirements $p > m \geq 1$ and $p\frac{m+1}{m} < 6$ from Assumption 2.1, we obtain the restriction that $1 < p < 5$ and $1 \leq m < 5$ for the validity of Theorem 2.13.

The second result is the blow up of potential well solutions with positive initial energy. Before stating this result, we shall define several constants. Let $y_0 > 0$ be the unique solution of the equation

$$MK_1(p+1)(2y_0)^{\frac{p-1}{2}} + MK_2(q+1)(2y_0)^{\frac{q-1}{2}} = 1. \tag{2.18}$$

The constants $0 < K_1, K_2 < \infty$ are given by

$$K_1 := \sup_{u \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}} \frac{\|u\|_{p+1}^{p+1}}{\|\nabla u\|_2^{p+1}}, \quad K_2 := \sup_{w \in H^2_0(\Gamma) \setminus \{0\}} \frac{|w|_{q+1}^{q+1}}{|\Delta w|_2^{q+1}}, \tag{2.19}$$

where K_1 and K_2 are well-defined when $1 \leq p \leq 5$ and $q \geq 1$. Also, we put

$$\hat{d} := y_0 - MK_1(2y_0)^{\frac{p+1}{2}} - MK_2(2y_0)^{\frac{q+1}{2}}, \tag{2.20}$$

where $M > 0$ has been introduced in (2.7).

Remark 2.15. We claim that

$$0 < \hat{d} \leq d, \tag{2.21}$$

where d is the depth of the potential well, defined in (2.15). The proof of inequality (2.21) can be found in the Appendix.

Also, we define the positive constant

$$A := \frac{\lambda}{2(6+\lambda)} y_0, \quad \text{where } \lambda := \min\{c_1 - 3, c_3 - 3\} > 0. \tag{2.22}$$

Then we have the following results.

Theorem 2.16 *(Blow-up with positive initial energy). Suppose that Assumption 2.1, Assumption 2.7 and Assumption 2.12 hold. Let*

$$E(0) > y_0, \quad \textit{and} \quad 0 \leq \mathcal{E}(0) < \min\{A, \hat{d}\}. \tag{2.23}$$

Then the weak solution $(u(t), w(t))$ of (1.1) blows up in finite time provided $p > m$ and $q > r$. In particular,

$$\limsup_{t \to T^-} (\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2) = +\infty,$$

for some $0 < T < \infty$.

Corollary 2.17. *Suppose that Assumption 2.1, Assumption 2.7 and Assumption 2.12 hold. Let $p > m$, $q > r$ and*

$$0 \leq \mathcal{E}(0) < \min\{A, \hat{d}\}.$$

If $(u_0, w_0) \in \mathcal{W}_2$, then the weak solution $(u(t), w(t))$ of (1.1) blows up in finite time.

3. Blow-up of solutions with negative initial energy

In this section, we prove Theorem 2.13, which says that the weak solution of system (1.1) blows up in finite time if the source terms are more dominant than damping terms and the initial total energy is negative.

Proof of Theorem 2.13. Let $(u(t), w(t)$ be a weak solution of (1.1) in the sense of Definition 2.4. We define the life span T of such a solution $(u(t), w(t))$ to be the supremum of all $T^* > 0$ such that $(u(t), w(t))$ is a solution to system (1.1) in the sense of Definition 2.4 on $[0, T^*]$. In the following, we will show that T is finite and obtain an upper bound for the life span of solutions.

As in [1,9,18], for any $t \in [0, T)$, we define

$$G(t) = -\mathcal{E}(t), \quad S(t) = \int_\Omega F(u(t))dx + \int_\Gamma H(w(t))d\Gamma,$$

where the total energy $\mathcal{E}(t)$ has been introduced in (2.9).

Clearly,

$$G(t) = -\frac{1}{2}(\|u_t\|_2^2 + |w_t|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 + |\Delta w|_2^2) + S(t),$$

which implies

$$\|u_t(t)\|_2^2 + |w_t(t)|_2^2 + \|\nabla u(t)\|_2^2 + \|u(t)\|_2^2 + |\Delta w(t)|_2^2 = -2G(t) + 2S(t). \tag{3.1}$$

We define

$$N(t) := \frac{1}{2}\left(\|u(t)\|_2^2 + |w(t)|_2^2\right) + \int_0^t \int_\Gamma \gamma u(\tau) \cdot w(\tau) d\Gamma d\tau, \tag{3.2}$$

then we have

$$N'(t) = \int_\Omega u(t)u_t(t)dx + \int_\Gamma w(t)w_t(t)d\Gamma + \int_\Gamma \gamma u(t) \cdot w(t) d\Gamma. \tag{3.3}$$

It follows from Assumption 2.12 that

$$S(t) \geq c_0\|u(t)\|_{p+1}^{p+1} + c_2|w(t)|_{q+1}^{q+1}. \tag{3.4}$$

Since $G(t) = -\mathcal{E}(t)$, the energy identity (2.10) can be written as

$$G(t) = G(0) + \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau.$$

Then from Assumption 2.1 and the regularity of (u, w), we infer that $G(t)$ is absolutely continuous, and

$$G'(t) = \int_\Omega g_1(u_t)u_t dx + \int_\Gamma g_2(w_t)w_t d\Gamma \geq \alpha\|u_t(t)\|_{m+1}^{m+1} + \alpha|w_t(t)|_{r+1}^{r+1} \geq 0, \tag{3.5}$$

a.e. on $[0, T)$. Then $G(t)$ is non-decreasing. In view of $G(0) = -\mathcal{E}(0) > 0$, we obtain that for any $0 \leq t < T$,

$$0 < G(0) \leq G(t) \leq S(t). \tag{3.6}$$

Due to (3.1) and (3.6), we obtain

$$\|u_t(t)\|_2^2 + |w_t(t)|_2^2 + \|\nabla u(t)\|_2^2 + \|u(t)\|_2^2 + |\Delta w(t)|_2^2 < 2S(t). \tag{3.7}$$

We introduce a constant a satisfying

$$0 < a < \min\left\{\frac{1}{m+1} - \frac{1}{p+1}, \ \frac{1}{r+1} - \frac{1}{q+1}, \ \frac{p-1}{2(p+1)}, \ \frac{q-1}{2(q+1)}\right\}. \tag{3.8}$$

Define

$$Y(t) := G^{1-a}(t) + \varepsilon N'(t), \tag{3.9}$$

where $0 < \varepsilon \le \min\{1, G(0)\}$ will be determined later. The function $Y(t)$ is adopted from the important work [16] by Georgiev and Todorova.

We aim to show that $Y(t)$ approaches infinity in finite time.

First we claim that

$$Y'(t) = (1-a)G^{-a}(t)G'(t) + \varepsilon N''(t), \tag{3.10}$$

where

$$\begin{aligned} N''(t) &= \|u_t(t)\|_2^2 + |w_t(t)|_2^2 - (\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2 + \|u(t)\|_2^2) - \int_\Omega g_1(u_t(t))u(t)dx \\ &\quad - \int_\Gamma g_2(w_t(t))w(t)d\Gamma + \int_\Omega u(t)f(u(t))dx + \int_\Gamma w(t)h(w(t))d\Gamma \\ &\quad + 2\int_\Gamma \gamma u(t)\cdot w_t(t)d\Gamma, \quad \text{a.e. on } [0,T). \end{aligned} \tag{3.11}$$

We remark that $N''(t)$ can be obtained *formally* by differentiating $N'(t)$ in (3.3) and using equations in (1.1). But this formal procedure needs to be justified as follows.

By Definition 2.4, $u_t \in L^{m+1}(\Omega\times(0,T))$. Since $u_0 \in H^1_{\Gamma_0}(\Omega) \hookrightarrow L^6(\Omega)$, then $u_0 \in L^{m+1}(\Omega)$ for $1 \le m < 5$ by referring to Remark 2.14. Then we have

$$\begin{aligned} \int_0^T\int_\Omega |u|^{m+1}dxdt &= \int_0^T\int_\Omega \left|\int_0^t u_t(\tau)d\tau + u_0\right|^{m+1} dxdt \\ &\le C(T^{m+1}\|u_t(t)\|^{m+1}_{L^{m+1}(\Omega\times(0,T))} + T\|u_0\|^{m+1}_{m+1}) < \infty. \end{aligned} \tag{3.12}$$

This implies $u(t) \in L^{m+1}(\Omega\times(0,T))$ for all $T \ge 0$. We can use the same argument to obtain $w(t) \in L^{r+1}(\Gamma\times(0,T))$. Then $u(t)$ and $w(t)$ enjoy the regularity restrictions imposed on the test functions $\phi(t)$ and $\psi(t)$, respectively, in Definition 2.4. Then we can replace ϕ by u in (2.2), ψ by w in (2.3) and use (3.3) to obtain

$$\begin{aligned} N'(t) &= (u,u_t)_\Omega + (w,w_t)_\Gamma + (\gamma u, w)_\Gamma \\ &= \int_\Omega u_0u_1dx + \int_\Gamma (w_0w_1 + \gamma u_0w_0)d\Gamma + \int_0^t (\|u_t\|_2^2 + |w_t|_2^2)d\tau \\ &\quad - \int_0^t (\|\nabla u\|_2^2 + |\Delta w|_2^2 + \|u\|_2^2)d\tau + 2\int_0^t\int_\Gamma \gamma u\cdot w_t d\Gamma d\tau - \int_0^t\int_\Omega g_1(u_t)udxd\tau \\ &\quad - \int_0^t\int_\Gamma g_2(w_t)wd\Gamma d\tau + \int_0^t\int_\Omega uf(u)dxd\tau + \int_0^t\int_\Gamma wh(w)d\Gamma d\tau. \end{aligned} \tag{3.13}$$

In the following, we show that $N'(t)$ is absolutely continuous, and therefore it can be differentiated.

Recall the fact $u \in C([0,t]; H^1_{\Gamma_0}(\Omega))$ and the embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^6(\Omega)$. By Remark 2.14, we know $1 < p < 5$. Hence, for all $t \in [0,T)$,

$$\int_0^t \left| \int_\Omega u f(u) dx \right| d\tau \le C \int_0^t \int_\Omega (|u|^p + 1)|u| dx d\tau < \infty. \tag{3.14}$$

Also, since $w \in H_0^2(\Gamma) \hookrightarrow L^\infty(\Gamma)$, we have

$$\int_0^t \left| \int_\Gamma w h(w) d\Gamma \right| d\tau \le C_T \int_0^t \int_\Gamma |w|^{q+1} d\Gamma d\tau < \infty. \tag{3.15}$$

By using the trace theorem, we see that

$$2\int_0^t \left| \int_\Gamma \gamma u \cdot w_t d\Gamma \right| d\tau \le C \int_0^t \|\nabla u\|_2^2 d\tau + \int_0^t |w_t|_2^2 d\tau < \infty. \tag{3.16}$$

Because of (3.12) and the regularity $u_t \in L^{m+1}(\Omega \times (0,T))$, we deduce that for all $t \in [0,T)$,

$$\int_0^t \left| \int_\Omega g_1(u_t) u dx \right| d\tau + \int_0^t \left| \int_\Gamma g_2(w_t) w d\Gamma \right| d\tau < \infty. \tag{3.17}$$

Then (3.14)-(3.17) and the regularity of (u,w) imply that all terms on the right-hand side of (3.13) are absolutely continuous, and thus we can differentiate (3.13) to conclude that the claimed formula (3.11) for $N''(t)$ holds true.

In the following, we aim to find a lower bound for $N''(t)$.

By using Young's inequality, we see that

$$2\left| \int_\Gamma \gamma u \cdot w_t \, d\Gamma \right| \le 2|w_t|_2^2 + \frac{1}{2}|\gamma u|_2^2. \tag{3.18}$$

Now we estimate the term $|\gamma u|_2^2$. Without loss of generality, we assume the flat portion Γ of the boundary is horizontal, and thus the unit normal vector to Γ is $\mathbf{n} = (0,0,1)$. Recall that $\partial\Omega = \overline{\Gamma_0 \cup \Gamma}$ and $u|_{\Gamma_0} = 0$. We define a vector field $\mathbf{F} = (0,0,u^2)$ and use the Divergence Theorem to get that

$$\begin{aligned} |\gamma u|_2^2 &= \int_\Gamma u^2 d\Gamma = \int_{\Gamma\cup\Gamma_0} u^2 d(\Gamma\cup\Gamma_0) = \int_{\Gamma\cup\Gamma_0} \mathbf{F}\cdot\mathbf{n}\, d(\Gamma\cup\Gamma_0) = \int_\Omega \operatorname{div}\mathbf{F}\, dx \\ &= \int_\Omega (u^2)_z dx = 2\int_\Omega u u_z dx \le \|u\|_2^2 + \|u_z\|_2^2 \le \|u\|_2^2 + \|\nabla u\|_2^2. \end{aligned} \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$2\left| \int_\Gamma \gamma u \cdot w_t d\Gamma \right| \le 2|w_t|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2. \tag{3.20}$$

Then (3.11) and (3.20) yield

$$N''(t) \geq \|u_t\|_2^2 - |w_t|_2^2 - \frac{3}{2}(\|\nabla u\|_2^2 + |\Delta w|_2^2 + \|u\|_2^2) - \int_\Omega g_1(u_t)udx - \int_\Gamma g_2(w_t)wd\Gamma + \int_\Omega uf(u)dx + \int_\Gamma wh(w)d\Gamma. \tag{3.21}$$

Noting $\|\nabla u\|_2^2 + |\Delta w|_2^2 + \|u\|_2^2 = -(\|u_t\|_2^2 + |w_t|_2^2) + 2S(t) - 2G(t)$ due to (3.1), and using the assumption $uf(u) \geq c_1 F(u)$, $wh(w) \geq c_3 H(w)$ from (2.16)-(2.17), we infer from (3.21) that

$$\begin{aligned} N''(t) &\geq \frac{5}{2}\|u_t\|_2^2 + \frac{1}{2}|w_t|_2^2 - 3S(t) + 3G(t) - \int_\Omega g_1(u_t)udx - \int_\Gamma g_2(w_t)wd\Gamma \\ &\quad + c_1\int_\Omega F(u)dx + c_3\int_\Gamma H(w)d\Gamma \\ &\geq \frac{5}{2}\|u_t\|_2^2 + \frac{1}{2}|w_t|_2^2 + 3G(t) - \int_\Omega g_1(u_t)udx - \int_\Gamma g_2(w_t)wd\Gamma + \lambda S(t), \end{aligned} \tag{3.22}$$

where we let $\lambda := \min\{c_1 - 3, c_3 - 3\} > 0$.

By using $g_1(s)s \leq \beta|s|^{m+1}$, Hölder's inequality and $p > m$, we have

$$\int_\Omega g_1(u_t)udx \leq \beta\int_\Omega |u_t|^m|u|dx \leq \beta\|u\|_{m+1}\|u_t\|_{m+1}^m \leq \beta|\Omega|^{\frac{p-m}{(p+1)(m+1)}}\|u\|_{p+1}\|u_t\|_{m+1}^m,$$

which along with (3.4) yields

$$\int_\Omega g_1(u_t)udx \leq \beta|\Omega|^{\frac{p-m}{(p+1)(m+1)}}c_0^{-\frac{1}{p+1}}S^{\frac{1}{p+1}}(t)\|u_t\|_{m+1}^m = R_1 S^{\frac{1}{p+1}}(t)\|u_t\|_{m+1}^m, \tag{3.23}$$

where the constant $R_1 := \beta|\Omega|^{\frac{p-m}{(p+1)(m+1)}}c_0^{-\frac{1}{p+1}}$.

Then by using Young's inequality, (3.5) and (3.6), we obtain from (3.23) that for any $\delta_1 > 0$,

$$\begin{aligned} \int_\Omega g_1(u_t)udx &\leq R_1 S^{\frac{1}{p+1}-\frac{1}{m+1}}(t)S^{\frac{1}{m+1}}(t)\|u_t\|_{m+1}^m \\ &\leq G^{\frac{1}{p+1}-\frac{1}{m+1}}(t)\left[\delta_1 S(t) + C_{\delta_1}R_1^{\frac{m+1}{m}}\|u_t\|_{m+1}^{m+1}\right] \\ &\leq \delta_1 G^{\frac{1}{p+1}-\frac{1}{m+1}}(t)S(t) + C_{\delta_1}\frac{R_1^{\frac{m+1}{m}}}{\alpha}G'(t)G^{-a}(t)G^{a+\frac{1}{p+1}-\frac{1}{m+1}}(t) \\ &\leq \delta_1 G^{\frac{1}{p+1}-\frac{1}{m+1}}(0)S(t) + C_{\delta_1}\frac{R_1^{\frac{m+1}{m}}}{\alpha}G'(t)G^{-a}(t)G^{a+\frac{1}{p+1}-\frac{1}{m+1}}(0), \end{aligned} \tag{3.24}$$

where $a > 0$ satisfying (3.8), and thus $a + \frac{1}{p+1} - \frac{1}{m+1} < 0$. Similarly, we can obtain for any $\delta_2 > 0$,

$$\int_\Gamma g_2(w_t)wd\Gamma \leq \delta_2 G^{\frac{1}{q+1}-\frac{1}{r+1}}(0)S(t) + C_{\delta_2}\frac{R_2^{\frac{r+1}{r}}}{\alpha}G'(t)G^{-a}(t)G^{a+\frac{1}{q+1}-\frac{1}{r+1}}(0), \tag{3.25}$$

where $R_2 := \beta|\Gamma|^{\frac{q-r}{(q+1)(r+1)}}c_2^{-\frac{1}{q+1}}$.

Inserting (3.24) and (3.25) into (3.22), we obtain

$$N''(t) \geq \frac{5}{2}\|u_t\|_2^2 + \frac{1}{2}|w_t|_2^2 + 3G(t) + \left[\lambda - \delta_1 G^{\frac{1}{p+1}-\frac{1}{m+1}}(0) - \delta_2 G^{\frac{1}{q+1}-\frac{1}{r+1}}(0)\right] S(t)$$
$$- \left[C_{\delta_1}\frac{R_1^{\frac{m+1}{m}}}{\alpha} G^{a+\frac{1}{p+1}-\frac{1}{m+1}}(0) + C_{\delta_2}\frac{R_2^{\frac{r+1}{r}}}{\alpha} G^{a+\frac{1}{q+1}-\frac{1}{r+1}}(0)\right] G'(t)G^{-a}(t). \tag{3.26}$$

Let us introduce the constants $\delta_1 = \frac{\lambda}{4}G^{\frac{1}{m+1}-\frac{1}{p+1}}(0)$ and $\delta_2 = \frac{\lambda}{4}G^{\frac{1}{r+1}-\frac{1}{q+1}}(0)$. Consequently, we infer from (3.10) and (3.26) that

$$Y'(t) \geq \left[(1-a) - \varepsilon C_{\delta_1}\frac{R_1^{\frac{m+1}{m}}}{\alpha} G^{a+\frac{1}{p+1}-\frac{1}{m+1}}(0) - \varepsilon C_{\delta_2}\frac{R_2^{\frac{r+1}{r}}}{\alpha} G^{a+\frac{1}{q+1}-\frac{1}{r+1}}(0)\right] G'(t)G^{-a}(t)$$
$$+ \frac{5}{2}\varepsilon\|u_t\|_2^2 + \frac{1}{2}\varepsilon|w_t|_2^2 + 3\varepsilon G(t) + \frac{\lambda}{2}\varepsilon S(t). \tag{3.27}$$

Noting that $0 < a < \frac{1}{2}$, we take $0 < \varepsilon < 1$ sufficiently small such that

$$\rho := (1-a) - \varepsilon C_{\delta_1}\frac{R_1^{\frac{m+1}{m}}}{\alpha} G^{a+\frac{1}{p+1}-\frac{1}{m+1}}(0) - \varepsilon C_{\delta_2}\frac{R_2^{\frac{r+1}{r}}}{\alpha} G^{a+\frac{1}{q+1}-\frac{1}{r+1}}(0) \geq 0,$$

to obtain from (3.27) that

$$Y'(t) \geq \rho G'(t)G^{-a}(t) + \frac{5}{2}\varepsilon\|u_t\|_2^2 + \frac{1}{2}\varepsilon|w_t|_2^2 + 3\varepsilon G(t) + \frac{\lambda}{2}\varepsilon S(t) > 0. \tag{3.28}$$

This shows that $Y(t)$ is increasing on $[0, T)$, with

$$Y(t) = G^{1-a}(t) + \varepsilon N'(t) > Y(0) = G^{1-a}(0) + \varepsilon N'(0).$$

If $N'(0) \geq 0$, then we do not need any further condition on ε. But, if $N'(0) < 0$, we further take ε such that $0 < \varepsilon \leq -\frac{G^{1-a}(0)}{2N'(0)}$. In any case, we have

$$Y(t) \geq \frac{1}{2}G^{1-a}(0) > 0, \quad \text{for } t \in [0, T). \tag{3.29}$$

Finally, we shall prove that the following inequality holds:

$$Y'(t) \geq C\varepsilon^{1+\sigma}Y^{\mu}(t), \quad \text{for } t \in [0, T), \tag{3.30}$$

where $C > 0$ is a generic constant independent of ε, and

$$1 < \mu = \frac{1}{1-a} < 2, \quad \sigma = \max\{\sigma_1, \sigma_2\} > 0,$$

and

$$\sigma_1 = 1 - \frac{2}{(1-2a)(p+1)} > 0, \quad \sigma_2 = 1 - \frac{2}{(1-2a)(q+1)} > 0,$$

due to (3.8).

Indeed, if $N'(t) \le 0$ for some $t \in [0,T)$, then for such value of t, we get

$$Y^\mu(t) = [G^{1-a}(t) + \varepsilon N'(t)]^\mu \le G(t). \tag{3.31}$$

Then we infer from (3.28) and (3.31) that

$$Y'(t) \ge 3\varepsilon G(t) \ge 3\varepsilon^{1+\sigma} G(t) \ge 3\varepsilon^{1+\sigma} Y^\mu(t).$$

If $N'(t) > 0$ for some $t \in [0,T)$, we first note that $Y(t) = G^{1-a}(t) + \varepsilon N'(t) \le G^{1-a}(t) + N'(t)$, then

$$Y^\mu(t) \le C\Big[G(t) + [N'(t)]^\mu\Big]. \tag{3.32}$$

Applying Hölder's inequality, Young's inequality, trace theorem and using $1 < \mu < 2$, we conclude from (3.3) that

$$\begin{aligned} [N'(t)]^\mu &\le \Big(\|u_t\|_2\|u\|_2 + |w_t|_2|w|_2 + |\gamma u|_2|w|_2\Big)^\mu \\ &\le C\Big(\|u_t\|_2^\mu\|u\|_2^\mu + |w_t|_2^\mu|w|_2^\mu + |\gamma u|_2^\mu|w|_2^\mu\Big) \\ &\le C\Big(\|u_t\|_2^2 + \|u\|_{p+1}^{\frac{2\mu}{2-\mu}} + |w_t|_2^2 + |w|_{q+1}^{\frac{2\mu}{2-\mu}} + \|\nabla u\|_2^2 + |w|_{q+1}^{\frac{2\mu}{2-\mu}}\Big). \end{aligned} \tag{3.33}$$

Since $\mu = \frac{1}{1-a}$ and $\sigma_1 > 0$, it follows that

$$\frac{2\mu}{(2-\mu)(p+1)} - 1 = \frac{2}{(1-2a)(p+1)} - 1 = -\sigma_1 < 0. \tag{3.34}$$

Noting $\varepsilon \le G(0)$, we infer from (3.4), (3.6) and (3.34) that

$$\begin{aligned} \|u(t)\|_{p+1}^{\frac{2\mu}{2-\mu}} &= (\|u(t)\|_{p+1}^{p+1})^{\frac{2\mu}{(2-\mu)(p+1)}} \le CS(t)^{\frac{2\mu}{(2-\mu)(p+1)}} \\ &\le CS(t)^{\frac{2\mu}{(2-\mu)(p+1)}-1} S(t) \le CG^{-\sigma_1}(0)S(t) \le C\varepsilon^{-\sigma_1} S(t). \end{aligned} \tag{3.35}$$

In the same way, we have

$$|w(t)|_{q+1}^{\frac{2\mu}{2-\mu}} \le C\varepsilon^{-\sigma_2} S(t). \tag{3.36}$$

Recall $\sigma = \max\{\sigma_1, \sigma_2\} > 0$ and $\varepsilon^{-\sigma} > 1$. By substituting (3.35) and (3.36) into (3.33), we get

$$\begin{aligned} [N'(t)]^\mu &\le C\Big(\|u_t\|_2^2 + |w_t|_2^2 + \|\nabla u\|_2^2 + \varepsilon^{-\sigma} S(t)\Big) \\ &\le C\Big(\|u_t\|_2^2 + |w_t|_2^2 + S(t) + \varepsilon^{-\sigma} S(t)\Big) \\ &\le C\varepsilon^{-\sigma}\Big(\|u_t\|_2^2 + |w_t|_2^2 + S(t)\Big), \end{aligned} \tag{3.37}$$

where (3.7) is used. Combining (3.28), (3.32) and (3.37), we derive that

$$\begin{aligned} Y'(t) &\ge C\varepsilon\Big[G(t) + \|u_t\|_2^2 + |w_t|_2^2 + S(t)\Big] \\ &\ge C\varepsilon\Big[G(t) + \varepsilon^\sigma [N'(t)]^\mu\Big] \\ &\ge C\varepsilon^{1+\sigma}\Big[G(t) + [N'(t)]^\mu\Big] \ge C\varepsilon^{1+\sigma} Y^\mu(t), \end{aligned}$$

for all values of $t \in [0, T)$ for which $N'(t) > 0$. Then in any case, (3.30) holds true.

It follows from (3.29) and (3.30) that the maximum life span T is necessarily finite with

$$T < C\varepsilon^{-(1+\sigma)} Y^{-\frac{a}{1-a}}(0) \leq C\varepsilon^{-(1+\sigma)} G^{-a}(0). \tag{3.38}$$

Notice that, at the blow-up time T, the quadratic energy must approach infinity:

$$\limsup_{t \to T^-} E(t) = +\infty. \tag{3.39}$$

We claim

$$\limsup_{t \to T^-} (\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2) = +\infty. \tag{3.40}$$

In fact, (3.1) shows that

$$E(t) = -G(t) + S(t) < S(t), \tag{3.41}$$

because $G(t) > 0$ on $[0, T)$. It follows from (3.40)-(3.41) that

$$\limsup_{t \to T^-} S(t) = +\infty. \tag{3.42}$$

Recall $p < 5$ from Remark 2.14, then we have

$$\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2 \geq C(\|u\|_{p+1}^{p+1} + |w|_{q+1}^{q+1}) \geq CS(t),$$

and along with (3.42), we obtain (3.40). The proof is completed. □

4. Blow-up of solutions with positive initial energy

This section is devoted to proving Theorem 2.16 and its corollary. These results state that the weak solution of system (1.1) blows up in finite time if the source terms dominate the damping terms, and the initial total energy $\mathcal{E}(0)$ is positive but sufficiently small, and the initial quadratic energy $E(0)$ is sufficiently large. The basic idea comes from the potential well theory.

4.1. Proof of Theorem 2.16

Proof of Theorem 2.16. We use some ideas from [16,21,30]. We define the life span T of such a solution $(u(t), w(t))$ to be the supremum of all $T^* > 0$ such that $(u(t), w(t))$ is a solution to system (1.1) in the sense of Definition 2.4 on $[0, T^*]$.

By using (2.7) and (2.19), we have that for $t \in [0, T)$,

$$\begin{aligned}
\mathcal{E}(t) &= E(t) - \int_\Omega F(u)dx - \int_\Gamma H(w)d\Gamma \\
&\geq E(t) - M\|u(t)\|_{p+1}^{p+1} - M|w(t)|_{q+1}^{q+1} \\
&\geq E(t) - MK_1\|\nabla u\|_2^{p+1} - MK_2|\Delta w|_2^{q+1} \\
&\geq E(t) - MK_1(2E(t))^{\frac{p+1}{2}} - MK_2(2E(t))^{\frac{q+1}{2}}, \quad \text{for all } t \in [0, T).
\end{aligned} \tag{4.1}$$

We define the function $F_1 : \mathbb{R}^+ \to \mathbb{R}$ by

$$F_1(y) := y - MK_1(2y)^{\frac{p+1}{2}} - MK_2(2y)^{\frac{q+1}{2}}, \tag{4.2}$$

where the positive constants K_1, K_2 were given in (2.19) and $M > 0$ was introduced in (2.7). Then (4.1) is equivalent to the following form:

$$\mathcal{E}(t) \geq F_1(E(t)), \quad \forall\, t \in [0, T). \tag{4.3}$$

In view of $p, q > 1$, we see that $F_1(y)$ is continuously differentiable, concave and has its maximum at $y = y_0 > 0$, where y_0 satisfies

$$MK_1(p+1)(2y_0)^{\frac{p-1}{2}} + MK_2(q+1)(2y_0)^{\frac{q-1}{2}} = 1. \tag{4.4}$$

We define

$$\hat{d} := \sup_{[0,\infty)} F_1(y) = F_1(y_0) = y_0 - MK_1(2y_0)^{\frac{p+1}{2}} - MK_2(2y_0)^{\frac{q+1}{2}}. \tag{4.5}$$

Since the function $F_1(y)$ has its maximum value at $y = y_0$, then $F_1(y)$ is decreasing if $y > y_0$. As $0 \leq \mathcal{E}(0) < \hat{d} = F_1(y_0)$, then there exists a unique constant y_1 such that

$$F_1(y_1) = \mathcal{E}(0), \quad \text{with } y_1 > y_0 > 0. \tag{4.6}$$

Then it follows from (4.3) that

$$\hat{d} = F_1(y_0) > F_1(y_1) = \mathcal{E}(0) \geq \mathcal{E}(t) \geq F_1(E(t)), \quad \forall\, t \in [0, T). \tag{4.7}$$

Note that $F_1(y)$ is continuous and decreasing if $y > y_0$, and $E(t)$ is also continuous. Since we assume $E(0) > y_0$, we infer from (4.7) that

$$E(t) \geq y_1 > y_0, \quad \forall\, t \in [0, T). \tag{4.8}$$

As in Section 3, we define

$$N(t) := \frac{1}{2}\left(\|u(t)\|_2^2 + |w(t)|_2^2\right) + \int_0^t \int_\Gamma \gamma u(\tau) \cdot w(\tau) d\Gamma d\tau,$$

and

$$S(t) := \int_\Omega F(u(t))dx + \int_\Gamma H(w(t))d\Gamma. \tag{4.9}$$

Let us define

$$\mathcal{G}(t) := A - \mathcal{E}(t), \tag{4.10}$$

where the constant A has been introduced in (2.22).

Due to the energy identity (2.10), we know $\mathcal{E}'(t) \leq 0$, and thus $\mathcal{G}'(t) \geq 0$, i.e., $\mathcal{G}(t)$ is non-decreasing in time. Since we assume $\mathcal{E}(0) < A$, then $\mathcal{G}(0) := A - \mathcal{E}(0) > 0$. Therefore, we have

$$\mathcal{G}(t) \geq \mathcal{G}(0) > 0, \quad \text{for all} \;\; t \in [0, T). \tag{4.11}$$

We consider the function

$$Y(t) := \mathcal{G}^{1-a}(t) + \varepsilon N'(t), \tag{4.12}$$

for some $a \in (0, \frac{1}{2})$ satisfying (3.8) and $\varepsilon > 0$. We plan to show that $Y(t)$ approaches infinity in finite time, by choosing ε sufficiently small. Adopting the same arguments as (3.10), we see that

$$Y'(t) = (1-a)\mathcal{G}^{-a}(t)\mathcal{G}'(t) + \varepsilon N''(t), \tag{4.13}$$

where

$$\begin{aligned} N''(t) = \|u_t(t)\|_2^2 + |w_t(t)|_2^2 - (\|\nabla u(t)\|_2^2 + \|u(t)\|_2^2 + |\Delta w(t)|_2^2) - \int\limits_{\Omega} g_1(u_t(t))u(t)dx \\ - \int\limits_{\Gamma} g_2(w_t(t))w(t)d\Gamma + \int\limits_{\Omega} u(t)f(u(t))dx + \int\limits_{\Gamma} w(t)h(w(t))d\Gamma \\ + 2\int\limits_{\Gamma} \gamma u(t) \cdot w_t(t)d\Gamma, \quad \text{a.e. on } [0, T). \end{aligned} \tag{4.14}$$

Following the same estimates as in (3.12)-(3.21), we obtain

$$\begin{aligned} N''(t) \geq \|u_t\|_2^2 - |w_t|_2^2 - \frac{3}{2}(\|\nabla u\|_2^2 + |\Delta w|_2^2 + \|u\|_2^2) - \int\limits_{\Omega} g_1(u_t)udx \\ - \int\limits_{\Gamma} g_2(w_t)wd\Gamma + \int\limits_{\Omega} uf(u)dx + \int\limits_{\Gamma} wh(w)d\Gamma. \end{aligned} \tag{4.15}$$

Since

$$\|\nabla u\|_2^2 + |\Delta w|_2^2 + \|u\|_2^2 = 2A - \|u_t\|_2^2 - |w_t|_2^2 + 2S(t) - 2\mathcal{G}(t),$$

and noting $uf(u) \geq c_1 F(u)$, $wh(w) \geq c_3 H(w)$, then we conclude from (4.15) that

$$N''(t) \geq \frac{5}{2}\|u_t\|_2^2 + \frac{1}{2}|w_t|_2^2 - 3A + 3\mathcal{G}(t) + \lambda S(t) - \int\limits_{\Omega} g_1(u_t)udx - \int\limits_{\Gamma} g_2(w_t)wd\Gamma,$$

where $\lambda := \min\{c_1 - 3, c_3 - 3\} > 0$. Because of (4.9), (4.10) and (2.9), we have

$$S(t) = \mathcal{G}(t) - A + E(t).$$

Then

$$\begin{aligned} N''(t) \geq \frac{5}{2}\|u_t\|_2^2 + \frac{1}{2}|w_t|_2^2 - \left(3 + \frac{\lambda}{2}\right)A + \left(3 + \frac{\lambda}{2}\right)\mathcal{G}(t) + \frac{\lambda}{2}E(t) + \frac{\lambda}{2}S(t) \\ - \int\limits_{\Omega} g_1(u_t)udx - \int\limits_{\Gamma} g_2(w_t)wd\Gamma, \quad \text{for } t \in [0, T). \end{aligned}$$

By recalling (4.8) and (2.22), one has

$$\frac{\lambda}{4}E(t) > \frac{\lambda}{4}y_0 = \left(3+\frac{\lambda}{2}\right)A, \quad \text{for all } \ t \in [0,T).$$

It follows that

$$N''(t) \geq \frac{5}{2}\|u_t\|_2^2 + \frac{1}{2}|w_t|_2^2 + \left(3+\frac{\lambda}{2}\right)\mathcal{G}(t) + \frac{\lambda}{4}E(t) + \frac{\lambda}{2}S(t) - \int_\Omega g_1(u_t)u dx - \int_\Gamma g_2(w_t)w d\Gamma, \quad \text{for } t \in [0,T). \tag{4.16}$$

Recalling $\lambda = \min\{c_1-3, c_3-3\} > 0$, we have $A = \frac{\lambda}{12+2\lambda}y_0 < y_0$. Then it is concluded from (4.8) that

$$\mathcal{G}(t) = A - \mathcal{E}(t) = A - E(t) + S(t) < y_0 - y_1 + S(t) < S(t), \tag{4.17}$$

for $t \in [0,T)$. Moreover, we infer from (4.10) and the energy inequality (2.10) that

$$\mathcal{G}'(t) = -\mathcal{E}'(t) = \int_\Omega g_1(u_t)u_t dx + \int_\Gamma g_2(w_t)w_t d\Gamma \geq \alpha\|u_t\|_{m+1}^{m+1} + \alpha|w_t|_{r+1}^{r+1} \geq 0, \tag{4.18}$$

for all $t \in [0,T)$. Then, we use the same arguments as in (3.24) and (3.25) to obtain from (4.17)-(4.18) that

$$\int_\Omega g_1(u_t)u dx \leq \delta_1 \mathcal{G}^{\frac{1}{p+1}-\frac{1}{m+1}}(0)S(t) + C_{\delta_1}\frac{R_1^{\frac{m+1}{m}}}{\alpha}\mathcal{G}'(t)\mathcal{G}^{-a}(t)\mathcal{G}^{a+\frac{1}{p+1}-\frac{1}{m+1}}(0), \tag{4.19}$$

$$\int_\Gamma g_2(w_t)w d\Gamma \leq \delta_2 \mathcal{G}^{\frac{1}{q+1}-\frac{1}{r+1}}(0)S(t) + C_{\delta_2}\frac{R_2^{\frac{r+1}{r}}}{\alpha}\mathcal{G}'(t)\mathcal{G}^{-a}(t)\mathcal{G}^{a+\frac{1}{q+1}-\frac{1}{r+1}}(0), \tag{4.20}$$

for any $\delta_1, \delta_2 > 0$, where the constant a satisfies (3.8).

Substituting (4.16), (4.19) and (4.20) into (4.13), we obtain

$$\begin{aligned} Y'(t) &= (1-a)\mathcal{G}^{-a}(t)\mathcal{G}'(t) + \varepsilon N''(t) \\ &\geq \left[(1-a) - \varepsilon C_{\delta_1}\frac{R_1^{\frac{m+1}{m}}}{\alpha}\mathcal{G}^{a+\frac{1}{p+1}-\frac{1}{m+1}}(0) - \varepsilon C_{\delta_2}\frac{R_2^{\frac{r+1}{r}}}{\alpha}\mathcal{G}^{a+\frac{1}{q+1}-\frac{1}{r+1}}(0)\right]\mathcal{G}^{-a}(t)\mathcal{G}'(t) \\ &\quad + \varepsilon\left[\frac{5}{2}\|u_t\|_2^2 + \frac{1}{2}|w_t|_2^2 + \left(3+\frac{\lambda}{2}\right)\mathcal{G}(t) + \frac{\lambda}{4}E(t)\right] \\ &\quad + \varepsilon\left[\frac{\lambda}{2} - \delta_1\mathcal{G}^{\frac{1}{p+1}-\frac{1}{m+1}}(0) - \delta_2\mathcal{G}^{\frac{1}{q+1}-\frac{1}{r+1}}(0)\right]S(t). \end{aligned} \tag{4.21}$$

At this point, we select $\delta_1, \delta_2 > 0$ such that

$$\frac{\lambda}{2} - \delta_1\mathcal{G}^{\frac{1}{p+1}-\frac{1}{m+1}}(0) - \delta_2\mathcal{G}^{\frac{1}{q+1}-\frac{1}{r+1}}(0) \geq \frac{\lambda}{4}.$$

For these fixed values of $\delta_1, \delta_2 > 0$, we choose $\varepsilon > 0$ sufficiently small that

$$(1-a) - \varepsilon C_{\delta_1}\frac{R_1^{\frac{m+1}{m}}}{\alpha}\mathcal{G}^{a+\frac{1}{p+1}-\frac{1}{m+1}}(0) - \varepsilon C_{\delta_2}\frac{R_2^{\frac{r+1}{r}}}{\alpha}\mathcal{G}^{a+\frac{1}{q+1}-\frac{1}{r+1}}(0) \geq \frac{1}{2}(1-a).$$

Then, from (4.21) we obtain

$$Y'(t) \geq \varepsilon \left[\frac{5}{2}\|u_t\|_2^2 + \frac{1}{2}|w_t|_2^2 + \left(3 + \frac{\lambda}{2}\right)\mathcal{G}(t) + \frac{\lambda}{4}E(t)\right] + \frac{\lambda}{4}\varepsilon S(t) > 0, \tag{4.22}$$

for all $t \in [0, T)$. Therefore, $Y(t)$ is increasing on $[0, T)$, with

$$Y(t) = \mathcal{G}^{1-a}(t) + \varepsilon N'(t) > Y(0) = \mathcal{G}^{1-a}(0) + \varepsilon N'(0).$$

Similar to (3.29), one can choose ε sufficiently small such that

$$Y(t) \geq \frac{1}{2}\mathcal{G}^{1-a}(0) > 0, \quad \text{for } t \in [0, T). \tag{4.23}$$

Now, we claim

$$Y'(t) \geq C\varepsilon^{1+\sigma}Y^{\mu}(t), \quad \text{for } t \in [0, T), \tag{4.24}$$

where $\mu := \frac{1}{1-a} \in (1, 2)$ and $\sigma := \max\{\sigma_1, \sigma_2\} > 0$ with $\sigma_1 = 1 - \frac{2}{(1-2a)(p+1)} > 0$ and $\sigma_2 = 1 - \frac{2}{(1-2a)(q+1)} > 0$. By solving differential inequality (4.24) with (4.23), we deduce that the maximum life span T is necessarily finite with

$$T < C\varepsilon^{-(1+\sigma)}Y^{-\frac{a}{1-a}}(0) \leq C\varepsilon^{-(1+\sigma)}\mathcal{G}^{-a}(0).$$

To prove (4.24), we use the following argument. If $N'(t) \leq 0$ for some $t \in [0, T)$, then for such value of t, we get

$$Y^{\mu}(t) = [\mathcal{G}^{1-a}(t) + \varepsilon N'(t)]^{\mu} \leq \mathcal{G}(t). \tag{4.25}$$

Then we infer from (4.22) and (4.25) that

$$Y'(t) \geq 3\varepsilon\mathcal{G}(t) \geq 3\varepsilon^{1+\sigma}\mathcal{G}(t) \geq 3\varepsilon^{1+\sigma}Y^{\mu}(t),$$

for any value of t such that $N'(t) \leq 0$.

If $N'(t) > 0$ for some $t \in [0, T)$, then

$$Y^{\mu}(t) \leq C\Big[\mathcal{G}(t) + [N'(t)]^{\mu}\Big]. \tag{4.26}$$

We know that $S(t) > \mathcal{G}(t) \geq \mathcal{G}(0) > 0$ by (4.17) and (4.11). Let $\varepsilon \leq \mathcal{G}(0)$. Then, following estimates (3.33)-(3.36), we can derive

$$[N'(t)]^{\mu} \leq C(\|u_t\|_2^2 + |w_t|_2^2 + \|\nabla u\|_2^2 + \varepsilon^{-\sigma}S(t)) \leq C\varepsilon^{-\sigma}(E(t) + S(t)). \tag{4.27}$$

Combining (4.22), (4.27) and (4.26), we arrive at

$$\begin{aligned} Y'(t) &\geq C\varepsilon\Big[\|u_t\|_2^2 + |w_t|_2^2 + \mathcal{G}(t) + E(t) + S(t)\Big] \\ &\geq C\varepsilon\Big[\mathcal{G}(t) + \varepsilon^{\sigma}[N'(t)]^{\mu}\Big] \geq C\varepsilon^{1+\sigma}\Big[\mathcal{G}(t) + [N'(t)]^{\mu}\Big] \geq C\varepsilon^{1+\sigma}Y^{\mu}(t), \end{aligned}$$

for any value of t such that $N'(t) > 0$. As a result, we conclude that (4.24) holds for all values of $t \in [0, T)$.

Finally, by using the same argument as in Section 3, we conclude $\limsup_{t\to T^-}(\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2) = +\infty$. This completes the proof. □

4.2. Proof of Corollary 2.17

Corollary 2.17 states that the weak solution of system (1.1) blows up in finite time if the source terms exceed the damping terms, and the initial total energy $\mathcal{E}(0)$ is positive but sufficiently small, and the initial data are from $\mathcal{W}_2$, i.e., the unstable part of the potential well.

Proof of Corollary 2.17. It suffices to show that if $(u_0, w_0) \in \mathcal{W}_2$, then $E(0) > y_0$.
Since $(u_0, w_0) \in \mathcal{W}_2$, then by the definition of $\mathcal{W}_2$, we get

$$\|\nabla u_0\|_2^2 + |\Delta w_0|_2^2 + \|u_0\|_2^2 < (p+1)\int\limits_{\Omega} F(u_0)dx + (q+1)\int\limits_{\Gamma} H(w_0)d\Gamma,$$

which together with (2.7) implies

$$\|\nabla u_0\|_2^2 + |\Delta w_0|_2^2 + \|u_0\|_2^2 < M(p+1)\|u_0\|_{p+1}^{p+1} + M(q+1)|w_0|_{q+1}^{q+1}. \tag{4.28}$$

Let $X := H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$ and recall the definition of K_1, K_2 in (2.19). Then we obtain from (4.28) that

$$\begin{aligned}\|(u_0, w_0)\|_X^2 &< M(p+1)K_1\|\nabla u_0\|_2^{p+1} + M(q+1)K_2|w_0|_2^{q+1} \\ &\le M(p+1)K_1\|(u_0, w_0)\|_X^{p+1} + M(q+1)K_2\|(u_0, w_0)\|_X^{q+1}.\end{aligned} \tag{4.29}$$

We divide both sides of (4.29) by $\|(u_0, w_0)\|_X^2$ to reach

$$M(p+1)K_1\|(u_0, w_0)\|_X^{p-1} + M(q+1)K_2\|(u_0, w_0)\|_X^{q-1} > 1.$$

This along with (2.18) gives

$$\begin{aligned}&MK_1(p+1)\left(\|(u_0, w_0)\|_X^2\right)^{\frac{p-1}{2}} + MK_2(q+1)\left(\|(u_0, w_0)\|_X^2\right)^{\frac{q-1}{2}} \\ &> 1 = MK_1(p+1)(2y_0)^{\frac{p-1}{2}} + MK_2(q+1)(2y_0)^{\frac{q-1}{2}}.\end{aligned}$$

Since $p, q > 1$, then we have

$$\|(u_0, w_0)\|_X^2 > 2y_0, \tag{4.30}$$

which implies that $E(0) > y_0$. Then, using Theorem 2.16, we obtain the blow-up of weak solutions in finite time. □

5. Appendix

5.1. Proof of Lemma 2.10

Proof of Lemma 2.10. Because of energy equality (2.10) and (2.12), we have

$$\mathcal{J}(u, w) \le \mathcal{E}(t) \le \mathcal{E}(0) < d, \quad \text{for } t \in [0, T). \tag{5.1}$$

Then $(u(t), w(t)) \in \mathcal{W}$. To prove $(u(t), w(t)) \in \mathcal{W}_2$, we argue by contradiction. We assume that there exists $t_1 \in (0, T)$ such that $(u(t_1), w(t_1)) \notin \mathcal{W}_2$. Recalling $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{W}$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$, then we obtain that $(u(t_1), w(t_1)) \in \mathcal{W}_1$.

By (2.8) and the mean value theorem, we can get that for any $t_0 \in [0, T)$,

$$\int_\Omega |F(u(t)) - F(u(t_0))| dx \le C \int_\Omega (|u(t)|^p + |u(t_0)|^p)|u(t) - u(t_0)| dx$$
$$\le C(\|u(t)\|_{p+1}^p + \|u(t_0)\|_{p+1}^p)\|u(t) - u(t_0)\|_{p+1}.$$

Noting $p \le 5$, and using the embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^6(\Omega)$ and the regularity of the weak solution $u \in C([0, T); H^1_{\Gamma_0}(\Omega))$, we conclude that $\int_\Omega F(u(t))dx \to \int_\Omega F(u(t_0))dx$ as $t \to t_0$, which implies that the function $t \mapsto \int_\Omega F(u(t))dx$ is continuous on $[0, T)$. Similarly, the continuity of the function $t \mapsto \int_\Gamma H(w(t))d\Gamma$ is obtained on $[0, T)$.

Since $(u(0), w(0)) \in \mathcal{W}_2$ and $(u(t_1), w(t_1)) \in \mathcal{W}_1$, then by the continuity and the intermediate value theorem, we know that there exists $s \in (0, t_1]$ such that

$$\|\nabla u(s)\|_2^2 + |\Delta w(s)|_2^2 + \|u(s)\|_2^2 = (p + 1) \int_\Omega F(u(s))dx + (q + 1) \int_\Gamma H(w(s))d\Gamma. \tag{5.2}$$

Define t^* be the infimum point over $s \in (0, t_1]$ satisfying (5.2). Then $t^* \in (0, t_1]$ and $(u(t), w(t)) \in \mathcal{W}_2$ for any $t \in [0, t^*)$. Two cases are considered as follows:

Case 1. $(u(t^*), w(t^*)) \neq (0, 0)$. Since (5.2) holds for t^*, then $(u(t^*), w(t^*)) \in \mathcal{N}$. We can get from (2.15) that $\mathcal{J}(u(t^*), w(t^*)) \ge d$. Since $\mathcal{E}(t) \ge \mathcal{J}(u(t), w(t))$ for any $t \in [0, T)$, $\mathcal{E}(t^*) \ge d$ is obtained. This contradicts (5.1).

Case 2. $(u(t^*), w(t^*)) = (0, 0)$. Note that $(u(t), w(t)) \in \mathcal{W}_2$ for any $t \in [0, t^*)$. We conclude from (2.7) that for any $t \in [0, t^*)$,

$$\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2 + \|u(t)\|_2^2 \le C(\|u(t)\|_{p+1}^{p+1} + |w(t)|_{q+1}^{q+1}) \le C(\|\nabla u(t)\|_2^{p+1} + |\Delta w(t)|_2^{q+1}),$$

which implies

$$\|(u(t), w(t))\|_X^2 < C(\|(u(t), w(t))\|_X^{p+1} + \|(u(t), w(t))\|_X^{q+1}), \quad t \in [0, t^*),$$

where $X = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$. Then, for any $t \in [0, t^*)$, we see that

$$\|(u(t), w(t))\|_X^{p-1} + \|(u(t), w(t))\|_X^{q-1} > \frac{1}{C}.$$

This gives us $\|(u(t), w(t))\|_X > s_0$, for any $t \in [0, t^*)$, where $s_0 > 0$ is the unique positive solution of $s^{p-1} + s^{q-1} = \frac{1}{C}$, where $p, q > 1$. It follows from the continuity of the weak solution $(u(t), w(t))$ that $\|(u(t^*), w(t^*))\|_X \ge s_0 > 0$. This contradicts that $(u(t^*), w(t^*)) = (0, 0)$. Therefore, $(u(t), w(t)) \in \mathcal{W}_2$ for all $t \in [0, T)$. □

5.2. Proof of inequality (2.21)

Proof of (2.21). We justify that $0 < \hat{d} \le d$. Indeed, by using (2.20) and (2.18), we have

$$\begin{aligned}
\hat{d} &= y_0 - MK_1(2y_0)^{\frac{p+1}{2}} - MK_2(2y_0)^{\frac{q+1}{2}} \\
&= y_0 - \frac{2y_0}{p+1} \cdot MK_1(p+1)(2y_0)^{\frac{p-1}{2}} - \frac{2y_0}{q+1} \cdot MK_2(q+1)(2y_0)^{\frac{q-1}{2}} \\
&\ge y_0 - \max\left\{\frac{2y_0}{p+1}, \frac{2y_0}{q+1}\right\} \left[MK_1(p+1)(2y_0)^{\frac{p-1}{2}} + MK_2(q+1)(2y_0)^{\frac{q-1}{2}}\right]
\end{aligned}$$

$$= y_0 - \max\left\{\frac{2y_0}{p+1}, \frac{2y_0}{q+1}\right\} = y_0 \cdot \min\left\{\frac{p-1}{p+1}, \frac{q-1}{q+1}\right\},$$

which, using the fact $p, q > 1$, implies $\hat{d} > 0$.

Let $X = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$. It follows from (2.7), (2.11) and (2.19) that

$$\begin{aligned}\mathcal{J}(u,w) &\geq \frac{1}{2}(\|\nabla u\|_2^2 + \|u\|_2^2 + |\Delta w|_2^2) - M(\|u\|_{p+1}^{p+1} + |w|_{q+1}^{q+1}) \\ &\geq \frac{1}{2}(\|\nabla u\|_2^2 + \|u\|_2^2 + |\Delta w|_2^2) - MK_1\|\nabla u\|_2^{p+1} - MK_2|\Delta w|_2^{q+1} \\ &\geq \frac{1}{2}\|(u,w)\|_X^2 - MK_1\|(u,w)\|_X^{p+1} - MK_2\|(u,w)\|_X^{q+1} \\ &:= \Lambda(\|(u,w)\|_X), \end{aligned} \tag{5.3}$$

with

$$\Lambda(y) = \frac{1}{2}y^2 - MK_1y^{p+1} - MK_2y^{q+1}.$$

Since $p, q > 1$, then

$$\Lambda'(y) = y[1 - MK_1(p+1)y^{p-1} - MK_2(q+1)y^{q-1}],$$

has only one positive zero at y^*, where y^* satisfies

$$MK_1(p+1)(y^*)^{p-1} + MK_2(q+1)(y^*)^{q-1} = 1. \tag{5.4}$$

It is easy to verify that $\Lambda(y)$ has maximum value at $y = y^*$, i.e.,

$$\Lambda(y^*) = \sup_{[0,\infty)} \Lambda(y) = \frac{1}{2}(y^*)^2 - MK_1(y^*)^{p+1} - MK_2(y^*)^{q+1}.$$

It follows from (2.18) and (5.4) that $(y^*)^2 = 2y_0$. Therefore,

$$\Lambda(y^*) = y_0 - MK_1(2y_0)^{\frac{p+1}{2}} - MK_2(2y_0)^{\frac{q+1}{2}} = \hat{d}. \tag{5.5}$$

From (5.3), we obtain

$$\mathcal{J}(\lambda(u,w)) \geq \Lambda(\lambda\|(u,w)\|_X), \text{ for all } \lambda \geq 0.$$

It follows that

$$\sup_{\lambda \geq 0} \mathcal{J}(\lambda(u,w)) \geq \Lambda(y^*).$$

Then we infer from (2.15) and (5.5) that

$$d = \inf_{(u,w)\in X\setminus(0,0)} \sup_{\lambda\geq 0} \mathcal{J}(\lambda(u,w)) \geq \Lambda(y^*) = \hat{d}.$$

This shows that $\hat{d}$ is not larger than the depth d of the potential well. $\square$

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