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Hadamard well-posedness for a hyperbolic equation of viscoelasticity with supercritical sources and damping

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Abstract

Presented here is a study of a viscoelastic wave equation with supercritical source and damping terms. We employ the theory of monotone operators and nonlinear semigroups, combined with energy methods to establish the existence of a unique local weak solution. In addition, it is shown that the solution depends continuously on the initial data and is global provided the damping dominates the source in an appropriate sense.
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1. Introduction

1.1. Literature overview

The theory of viscoelasticity encompasses description of materials that exhibit a combination of elastic (able to recover the original shape after stress application) and viscous (deformation-preserving after stress removal) characteristics. Quantitative description of such substances involves a strain–stress relation that depends on time. The classical linearized model yields an integro-differential equation that augments the associated elastic stress tensor with an appropriate *memory term* which encodes the history of the deformation gradient. The foundations of the theory go back to pioneering works of Boltzmann [10]. For fundamental modeling developments see [17] and the references therein.

When considering propagation of sound waves in viscoelastic fluids, if we neglect shear stresses then the stress tensor field may be expressed in terms of the acoustic pressure alone [31]. Thereby we obtain a scalar wave equation with a memory integral. This simplified formulation in fact captures most of the critical difficulties associated with the well-posedness of the viscoelastic vectorial model [17,30], and therefore the multi-dimensional scalar wave equation with memory will be the subject of the subsequent discussion.

One can consider such an integro-differential equation with a finite or infinite time delay (the former being a special case of the infinite-delay where the strain is zero for all $t < 0$). When restricted to the finite memory setting the system does not generate a semigroup, whereas the infinite-delay model can be represented as a semigroup evolution with the help of an appropriately defined history variable.

The (linear) viscoelastic problem with infinite memory and its stability were extensively addressed in [18,19,21]. Existence of global attractors for wave equations with infinite memory in the presence of nonlinear sources and linear internal damping (velocity feedback) was investigated in [30]. The "source" here refers to amplitude-dependent feedback nonlinearity whose growth rate is polynomially bounded with exponent $p \geq 1$. Due to the regularity of finite-energy solutions for this problem—H^1 Sobolev level for the displacement variable—the source considered in the latter reference was *subcritical* ($p < n/(n-2)$ for dimensions $n > 2$) with respect to this energy. Subsequently in [20] the authors look at attractors for the problem with strong (Kelvin–Voigt) damping and higher-order sources, including exponents of maximal order for which the associated energy is defined ($p = 5$ in 3D).

A larger body of work is available on the finite-time delay problem. The papers in this list focus predominantly on well-posedness and asymptotic stability with energy dissipation due to memory and interior and/or boundary velocity feedbacks. In addition, the sources, if present are at most critical, i.e., $p \leq n/(n-2)$ in dimensions above 2. See [13] for a treatment of interior and boundary memory with nonlinear boundary damping and no sources. Energy decay was investigated under localized interior dissipation and a source term was addressed in [14,15]. Local and global well-posedness with source, but now without additional frictional damping was the subject of the paper [3]. For systems of coupled waves with memory see [26]. Recent blow-up results for viscoelastic wave equations can be found in [27,28]. For quasilinear viscoelastic models with no sources and Kelvin–Voigt damping refer for example to [12,29].

Overall, it appears that the finite-time memory case has been more actively studied. Yet to our knowledge presently there are no works dealing with super-critical source exponents (i.e., $p > 3$ in 3D) in combination with memory terms. In light of this trend the present goal of this paper is two-fold:

- Analyze the viscoelastic wave equation with sources beyond the critical level—so the potential energy is no longer defined,—for instance, when $p > 5$ in 3-space dimensions. Our study is inspired by the advances in [5–8] and the consequent developments in [4,24,25,23,35].
- Provide a treatment of this problem in the context of evolution semigroup framework with a self-contained detailed description of the generator and function spaces.

1.2. The model

Throughout, $\Omega \subset \mathbb{R}^3$ is a bounded domain (open and connected) with boundary Γ of class C^2. Our results extend easily to bounded domains in $\mathbb{R}^n$, by accounting for the corresponding Sobolev embeddings, and accordingly adjusting the conditions imposed on the parameters. In this paper, we study the following model:

$$\begin{cases} u_{tt} - k(0)\Delta u - \int\limits_0^\infty k'(s)\Delta u(t-s)ds + g(u_t) = f(u), & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \Gamma \times \mathbb{R}, \\ u(x,t) = u_0(x,t), & \text{in } \Omega \times (-\infty, 0], \end{cases} \tag{1.1}$$

where, as mentioned earlier, the unknown u is an $\mathbb{R}$-valued function on $\Omega \times (0,\infty)$, which can be thought of as the acoustic pressure of sound waves in viscoelastic fluids. The differentiable scalar map k satisfies: $k(0), k(\infty) > 0$ with $k'(s) < 0$ for all $s > 0$. Here, g is a monotone feedback, and $f(u)$ is a source. The memory integral

$$\int\limits_0^\infty k'(s)\Delta u(t-s)ds$$

quantifies the viscous resistance and provides a weak form of energy dissipation. It also accounts for the full past history as time goes to $-\infty$, as opposed to the finite-memory model where the history is taken only over the interval $[0, t]$.

A similar model to (1.1) was studied in [30], but with a linear interior damping and a source of a *dissipative sign* which is at most sub-critical. In our model (1.1), the power-type damping $g(u_t)$ is nonlinear and not under any growth restrictions at the origin or at infinity; while the *energy building* source $f(u)$ is possibly of *supercritical* order.

For simplicity, we set $\mu(s) = -k'(s)$ and $k(\infty) = 1$, and so $k(0) > 1$. Thus, $\mu : \mathbb{R}^+ \to \mathbb{R}^+$, where in Assumption 1.1 below precise assumptions on μ will be imposed.

1.3. Notation

The following notations will be used throughout the paper:

$$\|u\|_s = \|u\|_{L^s(\Omega)}; \qquad \langle u, v \rangle = \langle u, v \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}; \qquad (u, v) = (u, v)_{L^2(\Omega)}. \tag{1.2}$$

The inner product on the weighted the Hilbert space $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$ is defined by

$$(u, v)_\mu := \int_0^\infty \int_\Omega \nabla u(s) \cdot \nabla v(s) dx \, \mu(s) ds. \tag{1.3}$$

Also, $\| \cdot \|_\mu$ represents the norm in $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$. Subsequently, we have:

$$H^1_\mu(\mathbb{R}^+, H^1_0(\Omega)) = \{u \in L^2_\mu(\mathbb{R}^+, H^1_0(\Omega)) : u_t \in L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))\}. \tag{1.4}$$

In particular, the space $L^2_\mu(\mathbb{R}^-, H^1_0(\Omega))$ consists of all functions $u : (-\infty, 0] \to H^1_0(\Omega)$ such that $u(-t) \in L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$. Thus,

$$H^1_\mu(\mathbb{R}^-, H^1_0(\Omega)) = \{u \in L^2_\mu(\mathbb{R}^-, H^1_0(\Omega)) : u_t \in L^2_\mu(\mathbb{R}^-, H^1_0(\Omega))\}.$$

1.4. Main results

In light of the above discussion, the following assumptions will be imposed throughout the paper.

Assumption 1.1.

- g is a continuous and monotone increasing feedback with $g(0) = 0$. In addition, the following growth condition at infinity holds: there exist positive constants a and b such that, for $|s| \geq 1$,

$$a|s|^{m+1} \leq g(s)s \leq b|s|^{m+1}, \quad \text{where } m \geq 1;$$

- $f \in C^1(\mathbb{R})$ such that $|f'(s)| \leq C(|s|^{p-1} + 1)$, $1 \leq p < 6$;
- $p\frac{m+1}{m} < 6$;
- $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ such that $\mu(s) > 0$ and $\mu'(s) \leq 0$ for all $s > 0$, and $\mu(\infty) = 0$;
- $u_0(x, t) \in L^2_\mu(\mathbb{R}^-, H^1_0(\Omega))$ with $\partial_t u_0(x, t) \in L^2_\mu(\mathbb{R}^-, L^2(\Omega))$ such that $u_0 : \mathbb{R}^- \to H^1_0(\Omega)$ and $\partial_t u_0(x, t) : \mathbb{R}^- \to L^2(\Omega)$ are weakly continuous at $t = 0$. In addition, for all $t \leq 0$, $u_0(x, t) = 0$ on Γ.

Let us note here that in view of the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ (in 3D), the Nemytski operator $f(u)$ is locally Lipschitz continuous from $H^1_0(\Omega)$ into $L^2(\Omega)$ for the values $1 \leq p \leq 3$. Hence, when the exponent of the sources p lies in $1 \leq p < 3$, we call the source *sub-critical*, and *critical*, if $p = 3$. For the values $3 < p \leq 5$ the source is called *supercritical*, and in this case the operator $f(u)$ is not locally Lipschitz continuous from $H^1_0(\Omega)$ into $L^2(\Omega)$. When $5 < p < 6$ the source is called *super-supercritical*. In this case, the potential energy may not be defined in the finite energy space and the problem itself is no longer within the framework of potential well theory.

Recently, the boundary value problem for the wave equation with nonlinear damping and supercritical source (but without the memory term):

$$\begin{cases} u_{tt} - \Delta u + g(u_t) = f(u), & \text{in } \Omega \times (0, \infty), \\ \partial_\nu u + u + g_0(u_t) = h(u), & \text{on } \Gamma \times (0, \infty), \end{cases}$$

has been studied in a series of papers [4–8]. One may see [9] for a summary of these results. Also, for other related work on nonlinear wave equations with supercritical sources, we refer the reader to [2,24,25,23,33–35].

It should be mentioned here that (1.1) is a monotonic problem well-suited for utilizing the theory of nonlinear semigroups and monotone operators (see for instance [1,36]). Thus, for the local well-posedness of (1.1), our strategy draws substantially from ideas in [5,8,23,35]. The essence of this strategy is to write the problem as a Cauchy problem of semigroup form and set up an appropriate phase space in order to verify the semigroup generator is m-accretive. The difficulty lies in the justification of the maximal monotonicity and coercivity of a certain nonlinear operator, which requires a correct choice of the function space and a combination of various techniques in monotone operator theory.

In order to state our main results, we begin with the definition of a weak solution of (1.1).

Definition 1.2. A function $u(x, t)$ is said to be a *weak solution* of (1.1) on $(-\infty, T]$ if $u \in L^2_\mu((-\infty, T]; H^1_0(\Omega)) \cap C([0, T]; H^1_0(\Omega))$ such that $u_t \in L^2_\mu((-\infty, T]; L^2(\Omega)) \cap C([0, T]; L^2(\Omega)) \cap L^{m+1}(\Omega \times (0, T))$ with:

- $u(x, t) = u_0(x, t)$ for $t \leq 0$;
- The following variational identity holds for all $t \in [0, T]$, and all test functions $\phi \in \mathscr{F}$:

$$
\big(u_t(t), \phi(t)\big) - \big(u_t(0), \phi(0)\big) - \int_0^t \int_\Omega u_t(\tau)\phi_t(\tau)dxd\tau + k(0)\int_0^t \int_\Omega \nabla u(\tau) \cdot \nabla \phi(\tau)dxd\tau
$$

$$
+ \int_0^t \int_0^\infty \int_\Omega \nabla u(\tau - s) \cdot \nabla \phi(\tau)dxk'(s)dsd\tau + \int_0^t \int_\Omega g\big(u_t(\tau)\big)\phi(\tau)dxd\tau
$$

$$
= \int_0^t \int_\Omega f\big(u(\tau)\big)\phi(\tau)dxd\tau, \tag{1.5}
$$

where

$$
\mathscr{F} = \big\{\phi \colon \phi \in C\big([0, T]; H^1_0(\Omega)\big) \cap L^{m+1}\big(\Omega \times (0, T)\big) \text{ with } \phi_t \in C\big([0, T]; L^2(\Omega)\big)\big\}.
$$

Our first theorem gives the existence and uniqueness of local weak solutions.

Theorem 1.3 (*Short-time existence*). *Assume the validity of Assumption 1.1, then there exists a local (in time) weak solution u to (1.1) defined on $(-\infty, T]$ for some $T > 0$ depending on the initial energy $E(0)$. Furthermore, the following energy identity holds:*

$$
E(t) + \int_0^t \int_\Omega g(u_t)u_t dxd\tau - \frac{1}{2}\int_0^t \int_0^\infty \big\|\nabla w(\tau, s)\big\|_2^2 \mu'(s)dsd\tau
$$

$$
= E(0) + \int_0^t \int_\Omega f(u)u_t dxd\tau, \tag{1.6}
$$

where $w(x, \tau, s) = u(x, \tau) - u(x, \tau - s)$, and the quadratic energy is defined by

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \int\limits_0^\infty \|\nabla w(t, s)\|_2^2 \mu(s)ds\right). \qquad (1.7)$$

Our next result states that weak solutions of (1.1) depend continuously on the initial data.

Theorem 1.4 *(Continuous dependence on initial data). In addition to Assumption 1.1, assume that $u_0(0) \in L^{\frac{3(p-1)}{2}}(\Omega)$ and $f \in C^2(\mathbb{R})$ such that $|f''(s)| \leq C(|s|^{p-2} + 1)$, for $p > 3$. If $u_0^n \in L_\mu^2(\mathbb{R}^-, H_0^1(\Omega))$ is a sequence of initial data such that $u_0^n \to u_0$ in $L_\mu^2(\mathbb{R}^-, H_0^1(\Omega))$ with $u_0^n(0) \to u_0(0)$ in $H_0^1(\Omega)$ and in $L^{\frac{3(p-1)}{2}}(\Omega)$, $\frac{d}{dt}u_0^n(0) \to \frac{d}{dt}u_0(0)$ in $L^2(\Omega)$, then the corresponding weak solutions u_n and u of (1.1) satisfy*

$$u_n \to u \quad in \ C\big([0, T]; H^1(\Omega)\big) \quad and \quad u_n' \to u' \quad in \ C\big([0, T]; L^2(\Omega)\big).$$

The uniqueness of weak solutions is a corollary of Theorem 1.4.

Corollary 1.5 *(Uniqueness). In addition to Assumption 1.1, we assume $u_0(0) \in L^{\frac{3(p-1)}{2}}(\Omega)$ and $f \in C^2(\mathbb{R})$ such that $|f''(s)| \leq C(|s|^{p-2} + 1)$, for $p > 3$. Then, weak solutions of (1.1) are unique.*

Our final result states: if the damping dominates the source term, then the solution is global. More precisely, we have

Theorem 1.6 *(Global existence). In addition to Assumption 1.1, further assume $u_0(0) \in L^{p+1}(\Omega)$. If $m \geq p$, then the weak solution of (1.1) is global.*

Remark 1.7. The classical condition that the "damping dominates the source," $m \geq p$, in Theorem 1.6 can be dispensed with if the source $f(u)$ in the equation satisfies suitable dissipativity conditions. For example, if the scalar function f is monotone decreasing with $f(s)s \leq 0$ for all $s \in \mathbb{R}$, then the assumption $m \geq p$ can be removed.

2. Local solutions

This section is devoted to prove the local existence statement in Theorem 1.3.

2.1. Operator theoretic formulation

In order to study the local solvability of (1.1), we exploit a remarkable idea due to Dafermos [18,19]: in addition to the displacement and velocity, we regard the past history of the displacement as a third variable. More precisely, introduce the *history function*:

$$w(x, t, s) = u(x, t) - u(x, t - s), \quad s \geq 0. \qquad (2.1)$$

After simple manipulations, problem (1.1) can be put into the following coupled system:

$$\begin{cases} u_t(x,t) = v(x,t) \\ v_t(x,t) = \Delta u(x,t) + \int_0^\infty \mu(s)\Delta w(x,t,s)ds - g\big(v(x,t)\big) + f\big(u(x,t)\big) \\ w_t(x,t,s) = v(x,t) - w_s(x,t,s), \end{cases} \qquad (2.2)$$

with boundary and initial conditions

$$\begin{cases} u(x,t) = 0 \quad \text{on } \Gamma \times [0,\infty), \\ w(x,t,s) = 0 \quad \text{on } \Gamma \times [0,\infty) \times [0,\infty) \\ u(x,0) = u_0(x,0) \\ v(x,0) = \dfrac{\partial u_0}{\partial t}(x,0) \\ w(x,0,s) = u_0(x,0) - u_0(x,-s) \\ w(x,t,0) = 0. \end{cases} \qquad (2.3)$$

Remark 2.1. System (2.2) with the given initial and boundary conditions is equivalent to the original system (1.1). In fact, by using the method of characteristics (in the (t,s)-plane where x is regarded as a fixed parameter), one can see that the equations $w_t = v - w_s$ and $u_t = v$ in (2.2) with the condition $w(x,t,0) = 0$ imply $w(x,t,s) = u(x,t) - u(x,t-s)$ for $s \geq 0$.

We establish the local in time existence of weak solutions in the so-called *past history framework*, i.e., the unknown function (u,v,w) is in the phase space

$$H := H_0^1(\Omega) \times L^2(\Omega) \times L_\mu^2\big(\mathbb{R}^+, H_0^1(\Omega)\big).$$

If $U = (u,v,w)$, $\hat{U} = (\hat{u}, \hat{v}, \hat{w}) \in H$, then the inner product on the Hilbert space H is the natural inner product given by:

$$(U, \hat{U})_H := \int_\Omega \nabla u \cdot \nabla \hat{u}\, dx + (v, \hat{v}) + (w, \hat{w})_\mu,$$

where $(v, \hat{v})$ and $(w, \hat{w})_\mu$, are given in (1.2)–(1.3).

If $\xi \in L_\mu^2(\mathbb{R}^+, H^2(\Omega) \cap H_0^1(\Omega))$, then clearly $\int_0^\infty \mu(s)\Delta\xi(s)ds \in L^2(\Omega) \subset H^{-1}(\Omega)$. Thus, for all $\phi \in H_0^1(\Omega)$, we have

$$\left\langle \int_0^\infty \mu(s)\Delta\xi(s)ds, \phi \right\rangle = \int_\Omega \left(\int_0^\infty \mu(s)\Delta\xi(s)ds \right)\phi dx$$

$$= -\int_0^\infty \int_\Omega \nabla\xi(s) \cdot \nabla\phi dx \mu(s)ds = -(\xi, \phi)_\mu, \qquad (2.4)$$

where $(\xi, \phi)_\mu$ is defined in (1.3).

Now, we define the operator $\mathcal{L} : D(\mathcal{L}) \subset L^2_\mu(\mathbb{R}^+, H^1_0(\Omega)) \to H^{-1}(\Omega)$ by

$$\mathcal{L}(\xi) = \int\limits_0^\infty \mu(s) \Delta \xi(s) ds$$

where $D(\mathcal{L}) = L^2_\mu(\mathbb{R}^+, H^2(\Omega) \cap H^1_0(\Omega))$. It follows from (2.4) that $\langle \mathcal{L}(\xi), \phi \rangle = -(\xi, \phi)_\mu$ for all $\phi \in H^1_0(\Omega)$. Clearly, $\mathcal{L}$ is a linear mapping, and in addition, $\mathcal{L}$ is bounded from $D(\mathcal{L})$ into $H^{-1}(\Omega)$. Indeed, for all $\xi \in D(\mathcal{L})$, we have

$$\big\| \mathcal{L}(\xi) \big\|_{H^{-1}(\Omega)} = \sup_{\|\phi\|_{H^1_0(\Omega)}=1} \big| (\xi, \phi)_\mu \big| \leq \|\xi\|_\mu \left(\int\limits_0^\infty \mu(s) ds \right)^{\frac{1}{2}} = \|\xi\|_\mu \big(k(0) - 1 \big).$$

Therefore, we can extend $\mathcal{L}$ to be a bounded linear operator (which is still denoted by $\mathcal{L}$) from $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$ to $H^{-1}(\Omega)$ such that, for any $\xi \in L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$,

$$\big\langle \mathcal{L}(\xi), \phi \big\rangle = -(\xi, \phi)_\mu \tag{2.5}$$

for all $\phi \in H^1_0(\Omega)$.

To this end, we define an (abstract) operator $\mathscr{A} : D(\mathscr{A}) \subset H \to H$ by

$$\mathscr{A}(U) = \left(\begin{array}{c} -v \\ -\Delta u + g(v) - \mathcal{L}(w) - f(u) \\ -v + w_s \end{array} \right)^{tr}$$

with its domain

$$D(\mathscr{A}) = \Big\{ (u, v, w) \in H^1_0(\Omega) \times H^1_0(\Omega) \times H^1_\mu\big(\mathbb{R}^+, H^1_0(\Omega)\big) : g(v) \in H^{-1}(\Omega) \cap L^1(\Omega),$$

$$- \Delta u + g(v) - \mathcal{L}(w) - f(u) \in L^2(\Omega), \ w(0) = 0 \Big\}.$$

Since the original w is a function of the three variables (x, t, s), then, in the definition of the operator $\mathscr{A}$ above, by saying $w \in H^1_\mu(\mathbb{R}^+, H^1_0(\Omega))$ we only mean the mapping: $\mathbb{R}^+ \ni s \mapsto w(\cdot, s)$ belongs to $H^1_\mu(\mathbb{R}^+, H^1_0(\Omega))$, as defined in (1.4).

Henceforth, system (2.2) can be reduced to the Cauchy problem:

$$\begin{cases} U_t + \mathscr{A}U = 0, \\ U(0) = U_0 = \big(u_0(x, 0), \partial_t u_0(x, 0), u_0(x, 0) - u_0(x, -s)\big). \end{cases} \tag{2.6}$$

2.2. Globally Lipschitz source

Our first proposition gives the existence of a global solution to the Cauchy problem (2.6) provided the source f is globally Lipschitz from $H^1_0(\Omega)$ to $L^2(\Omega)$.

Proposition 2.2. *Assume g is a continuous and monotone increasing function such that $g(0) = 0$. In addition, assume $f : H_0^1(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous. Then, system (2.6) has a unique global strong solution $U \in W^{1,\infty}(0, T; H)$ for arbitrary $T > 0$ provided the initial datum $U_0 \in D(\mathscr{A})$.*

Proof. In order to prove Proposition 2.2, it suffices to show that the operator $\mathscr{A} + \alpha I$ is m-accretive for some positive α. We say an operator $\mathscr{A} : \mathcal{D}(\mathscr{A}) \subset H \to H$ is *accretive* if $(\mathscr{A}x_1 - \mathscr{A}x_2, x_1 - x_2)_H \geq 0$, for all $x_1, x_2 \in \mathcal{D}(\mathscr{A})$, and it is *m-accretive* if, in addition, $\mathscr{A} + I$ maps $\mathcal{D}(\mathscr{A})$ onto H. It follows from Kato's Theorem (see [36] for instance) that, if $\mathscr{A} + \alpha I$ is m-accretive for some positive α, then for each $U_0 \in \mathcal{D}(\mathscr{A})$ there is a unique global strong solution U of the Cauchy problem (2.6).

Step 1. We show that $\mathscr{A} + \alpha I : D(\mathscr{A}) \subset H \to H$ is an accretive operator for some $\alpha > 0$. Let $U = (u, v, w)$, $\hat{U} = (\hat{u}, \hat{v}, \hat{w}) \in D(\mathscr{A})$. For sake of simplifying the notation in this proof, we use the notation $\langle \cdot, \cdot \rangle$ to denote the standard duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$; while $(\cdot, \cdot)$ represents the inner product in $L^2(\Omega)$.

Then,

$$
\begin{aligned}
\left((\mathscr{A} + \alpha I)U - (\mathscr{A} + \alpha I)\hat{U}, U - \hat{U}\right)_H &= \left(\mathscr{A}(U) - \mathscr{A}(\hat{U}), U - \hat{U}\right)_H + \alpha\|U - \hat{U}\|_H^2 \\
&= -\left(\nabla(v - \hat{v}), \nabla(u - \hat{u})\right) - \left\langle \Delta(u - \hat{u}), v - \hat{v}\right\rangle + \left\langle g(v) - g(\hat{v}), v - \hat{v}\right\rangle \\
&\quad - \left\langle \mathcal{L}(w - \hat{w}), v - \hat{v}\right\rangle - \left(f(u) - f(\hat{u}), v - \hat{v}\right) \\
&\quad - (v - \hat{v}, w - \hat{w})_\mu + (w_s - \hat{w}_s, w - \hat{w})_\mu + \alpha\|U - \hat{U}\|_H^2.
\end{aligned}
\tag{2.7}
$$

First, thanks to (2.5), we have

$$
-\left\langle \mathcal{L}(w - \hat{w}), v - \hat{v}\right\rangle = (w - \hat{w}, v - \hat{v})_\mu.
\tag{2.8}
$$

Since U and $\hat{U} \in D(\mathscr{A})$, we know $g(v) - g(\hat{v}) \in H^{-1}(\Omega) \cap L^1(\Omega)$. Thus, by the monotonicity of g and Lemma 2.6 in [1], one has $(g(v) - g(\hat{v}))(v - \hat{v}) \in L^1(\Omega)$ and

$$
\left\langle g(v) - g(\hat{v}), v - \hat{v}\right\rangle = \int_\Omega \left(g(v) - g(\hat{v})\right)(v - \hat{v})dx \geq 0.
\tag{2.9}
$$

Since $w - \hat{w} \in H_\mu^1(\mathbb{R}^+, H_0^1(\Omega))$, then by virtue of (1.3),

$$
\begin{aligned}
(w_s - \hat{w}_s, w - \hat{w})_\mu &= \frac{1}{2} \int_0^\infty \frac{d}{ds}\left(\int_\Omega |\nabla(w(s) - \hat{w}(s))|^2 dx\right)\mu(s)ds \\
&= -\frac{1}{2} \int_0^\infty \left(\int_\Omega |\nabla(w(s) - \hat{w}(s))|^2 dx\right)\mu'(s)ds \geq 0,
\end{aligned}
\tag{2.10}
$$

where we have used integration by parts and the facts: $\mu(\infty) = 0$, $\mu'(s) \leq 0$ and $w(0) = 0$.

Since f is globally Lipschitz continuous from $H_0^1(\Omega)$ into $L^2(\Omega)$ with Lipschitz constant L_f, it follows that

$$\big(f(u) - f(\hat{u}), v - \hat{v}\big) \leq L_f \|\nabla(u - \hat{u})\|_2 \|v - \hat{v}\|_2$$

$$\leq \frac{L_f}{2}\big(\|\nabla(u - \hat{u})\|_2^2 + \|v - \hat{v}\|_2^2\big). \tag{2.11}$$

Therefore, (2.7)–(2.11) yield

$$\big((\mathscr{A} + kI)U - (\mathscr{A} + kI)\hat{U}, U - \hat{U}\big)_H$$

$$\geq -\frac{L_f}{2}\big(\|u - \hat{u}\|_2^2 + \|v - \hat{v}\|_2^2\big) + \alpha\|U - \hat{U}\|_H^2$$

$$\geq \left(\alpha - \frac{L_f}{2}\right)\|U - \hat{U}\|_H^2 \geq 0; \tag{2.12}$$

provided $\alpha \geq \frac{L_f}{2}$.

Step 2. We show that $\mathscr{A} + \lambda I$ is m-accretive for some $\lambda > 0$. To this end, it suffices to show that the range of $\mathscr{A} + \lambda I$ is all of H, for some $\lambda > 0$ (see for example [36]).

Let $(a, b, c) \in H$. We aim to show that there exists $(u, v, w) \in D(\mathscr{A})$ such that $(\mathscr{A} + \lambda I) \times (u, v, w) = (a, b, c)$, for some $\lambda > 0$, i.e.,

$$\begin{cases} -v + \lambda u = a \\ -\Delta u + g(v) - \mathcal{L}(w) - f(u) + \lambda v = b \\ -v + w_s + \lambda w = c. \end{cases} \tag{2.13}$$

Notice that, (2.13) is equivalent to

$$\begin{cases} -\dfrac{1}{\lambda}\Delta v + g(v) - \mathcal{L}(w) - f\left(\dfrac{v + a}{\lambda}\right) + \lambda v = b + \dfrac{1}{\lambda}\Delta a \\ -v + w_s + \lambda w = c. \end{cases} \tag{2.14}$$

Let $X = H_0^1(\Omega) \times L_\mu^2(\mathbb{R}^+, H_0^1(\Omega))$ where X is endowed with the natural inner product, i.e., if $U = (v, w)$, $\hat{U} = (\hat{v}, \hat{w}) \in X$, then

$$(U, \hat{U})_X := \int_\Omega \nabla v \cdot \nabla \hat{v}\, dx + (w, \hat{w})_\mu.$$

Define an operator $T : D(T) \subset X \to X'$ by

$$T\begin{pmatrix} v \\ w \end{pmatrix}^{tr} = \begin{pmatrix} -\frac{1}{\lambda}\Delta v + g(v) - \mathcal{L}(w) - f(\frac{v+a}{\lambda}) + \lambda v \\ -v + w_s + \lambda w \end{pmatrix}^{tr}$$

where,

$$D(T) = \big\{(v, w) \in X : g(v) \in H^{-1}(\Omega) \cap L^1(\Omega),\ w \in H_\mu^1(\mathbb{R}^+, H_0^1(\Omega)),\ w(0) = 0\big\}.$$

It is important to note here that we consider $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$ as a Hilbert space identified with its own dual, and thus, $X' = H^{-1}(\Omega) \times L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$.

To justify the surjectivity of T, it is sufficient to show that the operator T is coercive and maximal monotone (Corollary 2.2 in [1]).

We split T as a summation of three operators. First, we define $T_1 : X \to X'$ by

$$T_1 \begin{pmatrix} v \\ w \end{pmatrix}^{tr} = \begin{pmatrix} -\frac{1}{\lambda}\Delta v - \mathcal{L}(w) - f(\frac{v+a}{\lambda}) + \lambda v \\ -v + \lambda w \end{pmatrix}^{tr}.$$

The operator $T_2 : D(T_2) \subset H^1_0(\Omega) \to H^{-1}(\Omega)$ is defined by

$$T_2(v) = g(v)$$

where $D(T_2) = \{v \in H^1_0(\Omega) : g(v) \in H^{-1}(\Omega) \cap L^1(\Omega)\}$. By a result due to Brézis [11], T_2 is the sub-differential of the convex functional $J : H^1_0(\Omega) \to [0, \infty]$ defined by $J(u) = \int_\Omega j(u)dx$, where $j(s) = \int_0^s g(\tau)d\tau$. It is well-known that the subdifferential of a proper convex function is maximal monotone, and thus T_2 is a maximal monotone operator.

We further define $T_3 : D(T_3) \subset L^2_\mu(\mathbb{R}^+, H^1_0(\Omega)) \to L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$ by

$$T_3(w) = \partial_s w$$

where $D(T_3) = \{w \in H^1_\mu(\mathbb{R}^+, H^1_0(\Omega)) : w(0) = 0\}$. Notice that the monotonicity of T_3 follows from (2.10). In addition, it is clear that the operator $T_3 + I$ is surjective. Therefore, T_3 is maximal monotone (see Theorem 2.2 in [1]).

To see T_1 is maximal monotone from X to X', it is enough to verify that T_1 is monotone and hemicontinuous. For checking the monotonicity of T_1, we consider

$$\left\langle T_1 \begin{pmatrix} v \\ w \end{pmatrix}^{tr} - T_1 \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix}^{tr}, \begin{pmatrix} v \\ w \end{pmatrix}^{tr} - \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix}^{tr} \right\rangle_{X' \times X}$$

$$= \frac{1}{\lambda}\|\nabla(v-\hat{v})\|_2^2 + (w-\hat{w}, v-\hat{v})_\mu - \left(f\left(\frac{v+a}{\lambda}\right) - f\left(\frac{\hat{v}+a}{\lambda}\right), v-\hat{v}\right)$$

$$+ \lambda\|v-\hat{v}\|_2^2 - (v-\hat{v}, w-\hat{w})_\mu + \lambda\|w-\hat{w}\|_\mu^2$$

$$= \frac{1}{\lambda}\|\nabla(v-\hat{v})\|_2^2 + \lambda\|v-\hat{v}\|_2^2 - \left(f\left(\frac{v+a}{\lambda}\right) - f\left(\frac{\hat{v}+a}{\lambda}\right), v-\hat{v}\right)$$

$$+ \lambda\|w-\hat{w}\|_\mu^2. \tag{2.15}$$

Since f is globally Lipschitz continuous from $H^1_0(\Omega)$ into $L^2(\Omega)$ with Lipschitz constant L_f, one has

$$\left(f\left(\frac{v+a}{\lambda}\right) - f\left(\frac{\hat{v}+a}{\lambda}\right), v-\hat{v}\right) \leq \left\|f\left(\frac{v+a}{\lambda}\right) - f\left(\frac{\hat{v}+a}{\lambda}\right)\right\|_2 \|v-\hat{v}\|_2$$

$$\leq \frac{L_f}{\lambda}\|\nabla(v-\hat{v})\|_2\|v-\hat{v}\|_2 \leq \frac{1}{2\lambda^2}\|\nabla(v-\hat{v})\|_2^2 + \frac{1}{2}L_f^2\|v-\hat{v}\|_2^2. \tag{2.16}$$

Combining (2.15) and (2.16) gives

$$
\left\langle T_1 \begin{pmatrix} v \\ w \end{pmatrix}^{tr} - T_1 \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix}^{tr}, \begin{pmatrix} v \\ w \end{pmatrix}^{tr} - \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix}^{tr} \right\rangle_{X' \times X}
$$
$$
\geq \left(\frac{1}{\lambda} - \frac{1}{2\lambda^2} \right) \|\nabla(v - \hat{v})\|_2^2 + \left(\lambda - \frac{1}{2} L_f^2 \right) \|v - \hat{v}\|_2^2 + \lambda \|w - \hat{w}\|_\mu^2. \tag{2.17}
$$

Thus, it follows from (2.17) that T_1 is strongly monotone; provided $\lambda > \frac{1}{2} \max\{L_f^2, 1\}$. Also, it is easy to see that strong monotonicity implies coercivity of T_1.

Next we verify $T_1 : X \to X'$ is hemicontinuous. Clearly, any linear operator is hemicontinuous. So, we merely consider the nonlinear term $f(\frac{v+a}{\lambda})$. The fact that f is globally Lipschitz from $H_0^1(\Omega)$ into $L^2(\Omega)$, trivially implies that $f(\frac{v+a}{\lambda})$ is continuous from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Hence, T_1 is hemicontinuous, and so, T_1 is maximal monotone.

Now, it is important to note that

$$
T \begin{pmatrix} v \\ w \end{pmatrix}^{tr} = T_1 \begin{pmatrix} v \\ w \end{pmatrix}^{tr} + \begin{pmatrix} T_2(v) \\ T_3(w) \end{pmatrix}^{tr},
$$

where T_2 and T_3 are both maximal monotone which act on different components of the vector $\begin{pmatrix} v \\ w \end{pmatrix}^{tr}$. By Proposition 7.1 in [23], it follows that the mapping $\begin{pmatrix} v \\ w \end{pmatrix}^{tr} \mapsto \begin{pmatrix} T_2(v) \\ T_3(w) \end{pmatrix}^{tr}$ is maximal monotone from $D(T)$ to X'. Moreover, due to the maximal monotonicity of T_1 and the fact that the domain of T_1 is the entire space X, we conclude that T is maximal monotone (Theorem 2.6 in [1]).

In addition, we know T is coercive, since T_1 is coercive and both of T_2 and T_3 are monotone. Therefore, T is maximal monotone and coercive, which yields the surjectivity of T, i.e., there exists $(v, w) \in D(T)$ that satisfies (2.14) for any $(a, b, c) \in H$. By (2.13), $u = \frac{v+a}{\lambda} \in H_0^1(\Omega)$ and $-\Delta u + g(v) - \mathcal{L}(w) - f(u) = b - \lambda v \in L^2(\Omega)$. Consequently, $(u, v, w) \in D(\mathscr{A})$ which concludes the proof of Proposition 2.2. $\square$

2.3. Locally Lipschitz source

In this subsection, we loosen the restriction on the source by allowing f to be locally Lipschitz continuous. More precisely, we have the following result.

Proposition 2.3. *Assume g is a continuous and monotone increasing function vanishing at the origin such that $g(s)s \geq a|s|^{m+1}$ for all $|s| \geq 1$, where $a > 0$ and $m \geq 1$. In addition, assume $f : H_0^1(\Omega) \to L^2(\Omega)$ is locally Lipschitz continuous. Then, system (2.6) has a unique local strong solution $U \in W^{1,\infty}(0, T; H)$, for some $T > 0$, provided the initial datum $U_0 \in D(\mathscr{A})$.*

Proof. We employ a standard truncation of the source. Define

$$
f_K(u) = \begin{cases} f(u) & \text{if } \|\nabla u\|_2 \leq K, \\ f(\frac{Ku}{\|\nabla u\|_2}) & \text{if } \|\nabla u\|_2 > K, \end{cases}
$$

where K is a positive constant. With this setting, we consider the following K-truncated problem:

$$U_t + \mathscr{A}_K U = 0 \tag{2.18}$$

with the same initial condition as in problem (2.6), where the operator $\mathscr{A}_K : D(\mathscr{A}_K) \subset H \to H$ is defined by

$$\mathscr{A}_K(U) = \begin{pmatrix} -v \\ -\Delta u + g(v) - \mathcal{L}(w) - f_K(u) \\ -v + w_s \end{pmatrix}^{tr}$$

with its domain $D(\mathscr{A}_K) = D(\mathscr{A})$.

Since the truncated source $f_K : H_0^1(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous for each K (see [16]), then by Proposition 2.2, the truncated problem (2.18) has a unique global strong solution $U_K \in W^{1,\infty}(0, T; H)$ for any $T > 0$; provided the initial datum $U_0 \in D(\mathscr{A})$.

For simplifying the notation in the rest of the proof, we shall express U_K as U. First, we aim to derive the associated energy identity for (2.18). Since $U = (u, v, w)$ is a strong solution of (2.18), the following equation holds:

$$v_t - \Delta u + g(v) - \mathcal{L}(w) - f_K(u) = 0, \quad \text{a.e. } [0, T]. \tag{2.19}$$

By the regularity of the solution U, we can multiply (2.19) by $v = u_t$ and integrate on $\Omega \times (0, t)$ where $0 < t < T$, to obtain,

$$\frac{1}{2}\left(\|v(t)\|_2^2 + \|\nabla u(t)\|_2^2\right) + \int_0^t \int_\Omega g(v)v\,dx\,d\tau + \int_0^t (w, v)_\mu\,d\tau$$

$$= \frac{1}{2}\left(\|v(0)\|_2^2 + \|\nabla u(0)\|_2^2\right) + \int_0^t \int_\Omega f_K(u)v\,dx\,d\tau, \tag{2.20}$$

where (2.20) holds for any $t > 0$, as $T > 0$ is arbitrary.

Since $v = w_t + w_s$, we compute

$$\int_0^t (w, v)_\mu\,d\tau = \int_0^t (w, w_t + w_s)_\mu\,d\tau$$

$$= \int_0^t \int_0^\infty \int_\Omega \nabla w(\tau, s) \cdot \nabla w_t(\tau, s)\,dx\,\mu(s)\,ds\,d\tau$$

$$+ \int_0^t \int_0^\infty \int_\Omega \nabla w(\tau, s) \cdot \nabla w_s(\tau, s)\,dx\,\mu(s)\,ds\,d\tau$$

$$= \frac{1}{2} \int_0^\infty \left(\|\nabla w(t,s)\|_2^2 - \|\nabla w(0,s)\|_2^2 \right) \mu(s)ds$$

$$- \frac{1}{2} \int_0^t \int_0^\infty \|\nabla w(\tau,s)\|_2^2 \mu'(s)ds d\tau, \tag{2.21}$$

where we have used integration by parts with the fact $\mu(\infty) = 0$ and $w(x,t,0) = 0$. Therefore, (2.20) and (2.21) yield the following energy identity:

$$E(t) + \int_0^t \int_\Omega g(v)v dx d\tau - \frac{1}{2} \int_0^t \int_0^\infty \|\nabla w(\tau,s)\|_2^2 \mu'(s)ds d\tau$$

$$= E(0) + \int_0^t \int_\Omega f_K(u)v dx d\tau, \tag{2.22}$$

where the quadratic energy $E(t)$ is defined in (1.7). Since $\mu'(s) \leq 0$, then for all $s > 0$, we have

$$E(t) + \int_0^t \int_\Omega g(v)v dx d\tau \leq E(0) + \int_0^t \int_\Omega f_K(u)v dx d\tau. \tag{2.23}$$

Let us note here that, straightforward calculation shows $f_K : H_0^1(\Omega) \to L^{\frac{m+1}{m}}(\Omega)$ is globally Lipschitz with Lipschitz constant L_K (see [16]). Thus, we estimate term due to the source on the right-hand side of the energy inequality (2.23) as follows:

$$\int_0^t \int_\Omega f_K(u)v dx d\tau \leq \int_0^t \|f_K(u)\|_{\frac{m+1}{m}} \|v\|_{m+1} d\tau$$

$$\leq \epsilon \int_0^t \|v\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \|f_K(u)\|_{\frac{m+1}{m}}^{\frac{m+1}{m}} d\tau$$

$$\leq \epsilon \int_0^t \|v\|_{m+1}^{m+1} d\tau + C_\epsilon L_K^{\frac{m+1}{m}} \int_0^t \|\nabla u\|_2^{\frac{m+1}{m}} d\tau + C_\epsilon t |f(0)|^{\frac{m+1}{m}} |\Omega|. \tag{2.24}$$

By recalling the assumption on the damping that $g(s)s \geq a|s|^{m+1}$ for all $|s| \geq 1$, we have

$$\int_0^t \int_\Omega g(v)v dx d\tau \geq a \int_0^t \|v\|_{m+1}^{m+1} d\tau - at|\Omega|. \tag{2.25}$$

Thus, (2.23)–(2.25) and the fact $\frac{m+1}{m} \leq 2$ yield,

$$E(t) + a \int_0^t \|v\|_{m+1}^{m+1} d\tau$$

$$\leq E(0) + \epsilon \int_0^t \|v\|_{m+1}^{m+1} d\tau + C_\epsilon L_K^{\frac{m+1}{m}} \int_0^t \|\nabla u\|_2^{\frac{m+1}{m}} d\tau + at|\Omega| + C_\epsilon t |f(0)|^{\frac{m+1}{m}} |\Omega|$$

$$\leq E(0) + \epsilon \int_0^t \|v\|_{m+1}^{m+1} d\tau + 2C_\epsilon L_K^{\frac{m+1}{m}} \int_0^t E(t) d\tau + t C_0 \tag{2.26}$$

where C_0 depends on ϵ, L_K, $f(0)$, $|\Omega|$ and m. By choosing $\epsilon \leq a$ one has,

$$E(t) \leq E(0) + C_0 T + C(L_K) \int_0^t E(\tau) d\tau, \quad \text{for all } t \in [0, T],$$

where $C(L_K) = 2C_\epsilon L_K^{\frac{m+1}{m}}$, and T will be chosen below. By Gronwall's inequality, one has

$$E(t) \leq \big(E(0) + C_0 T\big) e^{C(L_K)t}, \quad \text{for all } t \in [0, T].$$

If we select

$$T = \min\left\{ \frac{1}{C_0}, \frac{1}{C(L_K)} \log 2 \right\}, \tag{2.27}$$

then

$$E(t) \leq 2\big(E(0) + 1\big) \leq K^2/2, \quad \text{for all } t \in [0, T]; \tag{2.28}$$

provided we choose

$$K^2 \geq 4\big(E(0) + 1\big). \tag{2.29}$$

We note here that (2.28) shows $\|\nabla u\|_2 \leq K$, for all $t \in [0, T]$, and thus, by the definition of f_K, we see that $f_K(u) = f(u)$ on $[0, T]$. By the uniqueness of solutions, the solution of the truncated problem (2.18) coincides with the solution of the original problem (2.6) for $t \in [0, T]$. This completes the proof of Proposition 2.3. □

Remark 2.4. In Lemma 2.3, the local existence time T depends on L_K, which is the locally Lipschitz constant of $f : H_0^1(\Omega) \to L^2(\Omega)$; nevertheless, T is independent of the locally Lipschitz constant of $f : H_0^1(\Omega) \to L^{\frac{m+1}{m}}(\Omega)$. This observation is crucial for the next step.

2.4. Completion of the proof for the local existence

To extend the existence result in Proposition 2.3 to the situation where the source $f(u)$ is not necessary locally Lipschitz from $H_0^1(\Omega)$ into $L^2(\Omega)$, we employ the following truncation of the source (first used in [32]). Namely, put:

$$f_n(u) = f(u)\eta_n(u) \tag{2.30}$$

where $\eta_n \in C_0^\infty(\mathbb{R})$ is a smooth cutoff function such that: $0 \leq \eta_n \leq 1$; $\eta_n(u) = 1$ if $|u| \leq n$; $\eta_n(u) = 0$ if $|u| \geq 2n$; and $|\eta'(u)| \leq C/n$.

The following result is already known in [8,35].

Proposition 2.5. *Suppose* $m \geq 1$, $0 < \epsilon < 1$. *Assume* $f : \mathbb{R} \to \mathbb{R}$ *such that* $|f'(s)| \leq C(|s|^{p-1} + 1)$, *where* $p\frac{m+1}{m} \leq \frac{6}{1+2\epsilon}$. *Let* f_n *be defined in (2.30). Then,*

- $f_n : H_0^1(\Omega) \to L^2(\Omega)$ *is globally Lipschitz continuous with Lipschitz constant depending on* n.
- $f_n : H_0^{1-\epsilon}(\Omega) \to L^{\frac{m+1}{m}}(\Omega)$ *is locally Lipschitz continuous with a local Lipschitz constant independent of* n.

We are now ready to prove the local existence statement in Theorem 1.3.

Proof. By using the truncated source f_n defined in (2.30), we define the nonlinear operator

$$\mathscr{A}_n(U) = \begin{pmatrix} -v \\ -\Delta u + g(v) - \mathcal{L}(w) - f_n(u) \\ -v + w_s \end{pmatrix}$$

with its domain

$$D(\mathscr{A}_n) = \Big\{ (u, v, w) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H_\mu^1\big(\mathbb{R}^+, H_0^1(\Omega)\big) : g(v) \in H^{-1}(\Omega) \cap L^1(\Omega),$$
$$-\Delta u + g(v) - \mathcal{L}(w) - f_n(u) \in L^2(\Omega), \ w(0) = 0 \Big\}.$$

By Proposition 2.5, $f_n(u) \in L^2(\Omega)$ for all $u \in H_0^1(\Omega)$, so $D(\mathscr{A}_n)$ is uniform for all n. For the initial data $u_0(x, t) \in L_\mu^2(\mathbb{R}^-, H_0^1(\Omega))$ satisfying Assumption 1.1, there exists $u_0^n(x, t) \in H_\mu^1(\mathbb{R}^-, C_0^2(\Omega))$, $n \in \mathbb{N}$, such that $u_0^n(x, t) \to u_0(x, t)$ in $L_\mu^2(\mathbb{R}^-, H_0^1(\Omega))$ with $u_0^n(x, 0) \to u_0(x, 0)$ in $H_0^1(\Omega)$ and $v_0^n(x, 0) \to v_0(x, 0)$ in $L^2(\Omega)$, where $v_0^n = \frac{d}{dt}u_0^n$ and $v_0 = \frac{d}{dt}u_0$. Put $w_0^n(x, s) = u_0^n(x, 0) - u_0^n(x, -s)$. Notice, $U_0^n := (u_0^n(x, 0), v_0^n(x, 0), w_0^n(x, s)) \in D(\mathscr{A}_n)$, for every $n \in \mathbb{N}$. Therefore, by Propositions 2.3 and 2.5, the approximate system

$$U_t + \mathscr{A}_n U = 0 \tag{2.31}$$

with the initial data U_0^n has a unique local strong solution $U_n = (u_n, v_n, w_n) \in W^{1,\infty}(0, T; H)$. Thanks to Proposition 2.5, the life span T of each solution U_n, given in (2.27), is independent of n, since the local Lipschitz constant of the mapping $f_n : H^1(\Omega) \to L^{\frac{m+1}{m}}(\Omega)$ is independent of n. Also, we known that T depends on K, where $K^2 \geq 4(E(0) + 1)$; nonetheless, since

$E_n(0) \to E(0)$, we can choose K sufficiently large so that K is independent of n. Now, by (2.28) one has $E_n(t) \leq K^2/2$, which implies the uniform boundedness of $\|U_n(t)\|_H$ on $[0, T]$. More precisely, we have

$$\left\|U_n(t)\right\|_H^2 = \left\|\nabla u_n(t)\right\|_2^2 + \left\|u_n'(t)\right\|_2^2 + \int_0^\infty \left\|\nabla w_n(t, s)\right\|_2^2 \mu(s) ds \leq K^2 \qquad (2.32)$$

for all $t \in [0, T]$ and all $n \in \mathbb{N}$. By choosing $\epsilon \leq a/2$ in (2.26) and by the fact $E_n(t)$ is uniformly bounded on $[0, T]$, one has

$$\int_0^T \left\|u_n'\right\|_{m+1}^{m+1} dt \leq C_K \qquad (2.33)$$

for some constant $C_K > 0$ depending on K. In addition, by Remark 2.1, one has $w_n(x, t, s) = u_n(x, t) - u_n(x, t - s)$, for all $t, s \geq 0$.

It follows from (2.32) and (2.33) that there exists $U = (u, v, w) \in L^\infty(0, T; H)$ such that, on a subsequence,

$$U_n \to U \quad \text{weak* in } L^\infty(0, T; H) \qquad (2.34)$$

and

$$u_n' \to u' \quad \text{weakly in } L^{m+1}\big(\Omega \times (0, T)\big). \qquad (2.35)$$

Also, it is straightforward to show that $v = u'$ and $w(x, t, s) = u(x, t) - u(x, t - s)$ for a.e. $t, s \geq 0$.

By virtue of (2.32) and (2.34), we infer

$$E(t) = \frac{1}{2}\left(\left\|\nabla u(t)\right\|_2^2 + \left\|u'(t)\right\|_2^2 + \int_0^\infty \left\|\nabla w(t, s)\right\|_2^2 \mu(s) ds\right)$$

$$= \frac{1}{2}\left\|U(t)\right\|_H^2 \leq \frac{K^2}{2} \qquad (2.36)$$

for all $t \in [0, T]$. Similarly, from (2.33) and (2.35), it follows that

$$\int_0^T \left\|u'\right\|_{m+1}^{m+1} dt \leq C_K. \qquad (2.37)$$

Furthermore, by using (2.34) and Aubin's compactness theorem (see for instance [37]), we infer

$$u_n \to u \quad \text{strongly in } L^\infty\big(0, T; H^{1-\epsilon}(\Omega)\big), \qquad (2.38)$$

for $0 < \epsilon < 1$. Since $U_n \in D(\mathscr{A}_n)$ is a strong solution of (2.31), the following variational formula holds:

$$\big(u_n'(t), \phi(t)\big)_\Omega - \big(u_n'(0), \phi(0)\big)_\Omega - \int_0^t \int_\Omega u_n'(\tau)\phi'(\tau)dxd\tau + \int_0^t \int_\Omega \nabla u_n(\tau) \cdot \nabla \phi(\tau)dxd\tau$$

$$+ \int_0^t \int_\Omega g\big(u_n'(\tau)\big)\phi(\tau)dxd\tau + \int_0^t \int_0^\infty \int_\Omega \nabla w_n(\tau, s) \cdot \nabla \phi(\tau)dx\mu(s)dsd\tau$$

$$= \int_0^t \int_\Omega f_n\big(u_n(\tau)\big)\phi(\tau)dxd\tau. \tag{2.39}$$

for all $\phi \in C([0, T]; H_0^1(\Omega)) \cap L^{m+1}(\Omega \times (0, T))$ with $\phi_t \in C([0, T]; L^2(\Omega))$ and for a.e. $t \in [0, T]$.

Now, we fix an arbitrary $t \in [0, T]$, and show the convergence of nonlinear terms in (2.39). We shall first show:

$$\lim_{n \to \infty} \int_0^t \int_\Omega f_n(u_n)\phi dxd\tau = \int_0^t \int_\Omega f(u)\phi dxd\tau, \tag{2.40}$$

for all $\phi \in C([0, T]; H_0^1(\Omega)) \cap L^{m+1}(\Omega \times (0, T))$ and a.e. $t \in [0, T]$. Indeed, we have

$$\left| \int_0^t \int_\Omega \big(f_n(u_n) - f(u)\big)\phi dxd\tau \right|$$

$$\leq \int_0^t \int_\Omega \big| f_n(u_n) - f_n(u)\big| |\phi| dxd\tau + \int_0^t \int_\Omega \big| f_n(u) - f(u)\big| |\phi| dxd\tau. \tag{2.41}$$

We shall estimate each term on the right hand side of (2.41). By recalling Proposition 2.5, we know that f_n is locally Lipschitz continuous from $H^{1-\epsilon}(\Omega) \to L^{\frac{m+1}{m}}(\Omega)$ with the Lipschitz constant independent of n. Thus,

$$\int_0^t \int_\Omega \big| f_n(u_n) - f_n(u)\big| |\phi| dxd\tau$$

$$\leq \left(\int_0^t \int_\Omega \big| f_n(u_n) - f_n(u)\big|^{\frac{m+1}{m}} dxd\tau \right)^{\frac{m}{m+1}} \left(\int_0^t \int_\Omega |\phi|^{m+1} dxd\tau \right)^{\frac{1}{m+1}}$$

$$\leq C(K)\|\phi\|_{L^{m+1}(\Omega \times (0,T))} \left(\int_0^t \|u_n - u\|_{H^{1-\epsilon}(\Omega)}^{\frac{m+1}{m}} d\tau \right)^{\frac{m}{m+1}} \to 0, \tag{2.42}$$

where we have used the strong convergence (2.38).

The second term on the right hand side of (2.41) is handled as follows:

$$
\int_0^t \int_\Omega |f_n(u) - f(u)||\phi| dx d\tau
$$

$$
\leq \left(\int_0^t \int_\Omega |f_n(u) - f(u)|^{\frac{m+1}{m}} dx d\tau \right)^{\frac{m}{m+1}} \left(\int_0^t \int_\Omega |\phi|^{m+1} dx d\tau \right)^{\frac{1}{m+1}}
$$

$$
\leq \|\phi\|_{L^{m+1}(\Omega \times (0,T))} \left(\int_0^t \int_\Omega |f(u)|^{\frac{m+1}{m}} |\eta_n(u) - 1|^{\frac{m+1}{m}} dx d\tau \right)^{\frac{m}{m+1}}. \qquad (2.43)
$$

Thanks to the assumptions $|f(u)| \leq C(|u|^p + 1)$, $p\frac{m+1}{m} < 6$, and the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, it can be easily shown that $f(u) \in L^{\frac{m+1}{m}}(\Omega \times (0,T))$, for each $u \in H_0^1(\Omega)$. Also, notice $\eta_n(u(x)) \to 1$ a.e. in Ω. Thus, by the Lebesgue Dominated Convergence Theorem, it follows that the right hand side of (2.43) converges to zero, and along with (2.42), we conclude the right hand side of (2.41) is also convergent to zero. Therefore, (2.40) follows.

In order to deal with the term due to damping in (2.39), we recall the assumption $g(s)s \leq |s|^{m+1}$ for all $|s| \geq 1$, then by (2.33) one has $g(u_n')$ is uniformly bounded in the space $L^{\frac{m+1}{m}}(\Omega \times (0,t))$. Therefore, there exists $g^* \in L^{\frac{m+1}{m}}(\Omega \times (0,t))$ such that, on a subsequence

$$
g(u_n') \to g^* \quad \text{weakly in } L^{\frac{m+1}{m}}(\Omega \times (0,t)). \qquad (2.44)
$$

We aim to show $g^* = g(u')$. To accomplish this assertion, we consider two solutions U_n and U_j of the approximate problem (2.31) corresponding to the parameters n and j, respectively. By denoting $\tilde{U} = U_n - U_j$, then the following energy inequality holds:

$$
\tilde{E}(t) + \int_0^t \int_\Omega \left(g(u_n') - g(u_j') \right) \tilde{u}' dx d\tau \leq \tilde{E}(0) + \int_0^t \int_\Omega |f_n(u_n) - f_j(u_j)||\tilde{u}'| dx d\tau, \qquad (2.45)
$$

where $\tilde{E}(t) = \frac{1}{2}(\|\tilde{u}'(t)\|_2^2 + \|\nabla\tilde{u}(t)\|_2^2 + \int_0^\infty \|\nabla\tilde{w}(t,s)\|_2^2 \mu(s) ds)$.

Let us show first that $\int_0^t \int_\Omega |f_n(u_n) - f_j(u_j)|\tilde{u}' dx d\tau \to 0$ as $n, j \to \infty$. Indeed,

$$
\int_0^t \int_\Omega |f_n(u_n) - f_j(u_j)||\tilde{u}'| dx d\tau
$$

$$
\leq \int_0^t \int_\Omega |f_n(u_n) - f_n(u)||\tilde{u}'| dx d\tau + \int_0^t \int_\Omega |f_n(u) - f(u)||\tilde{u}'| dx d\tau
$$

$$
+ \int_0^t \int_\Omega |f(u) - f_j(u)||\tilde{u}'| dx d\tau + \int_0^t \int_\Omega |f_j(u) - f_j(u_j)||\tilde{u}'| dx d\tau. \qquad (2.46)
$$

By replacing ϕ by $\tilde{u}'$ in (2.42) and (2.43), we conclude that the right hand side of (2.46) converges to zero, and along with the fact $U_0^n \to U_0$ in H, it follows that the right hand side of (2.45) converges to zero. Thus,

$$\int_0^t \big(g(u_n') - g(u_j')\big)(u_n' - u_j')dxd\tau \to 0 \tag{2.47}$$

as $n, j \to \infty$. Note, since the function g is increasing, it is straightforward to show the operator $g(\cdot) : L^{m+1}(\Omega \times (0, t)) \to L^{\frac{m+1}{m}}(\Omega \times (0, t))$ is monotone and hemicontinuous, which implies it is maximal monotone. Thus, with this in hand, it follows from (2.44) and (2.47) that (see Lemma 2.3 in [1]), $g^* = g(u')$. Hence,

$$g(u_n') \to g(u') \quad \text{weakly in } L^{\frac{m+1}{m}}(\Omega \times (0, t)). \tag{2.48}$$

Now, by (2.34), (2.40) and (2.48), we can pass to limit on (2.39), and conclude that (1.5) holds.

It remains to show the continuity of the solution, and verify that u satisfies the initial condition. Indeed, since g is a increasing function, then (2.45) yields

$$\tilde{E}(t) \leq \tilde{E}(0) + \int_0^T \int_\Omega \big|f_n(u_n) - f_j(u_j)\big|\big|\tilde{u}'\big|dxd\tau, \quad \text{for all } t \in [0, T]. \tag{2.49}$$

Notice that the right hand side of (2.49) does not depend on the value of t and it does converge to zero as $n, j \to \infty$. Therefore, it follows that

$$u_n \to u \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad u_n' \to u' \quad \text{in } L^2(\Omega), \quad \text{uniformly on } [0, T]. \tag{2.50}$$

Since $u_n \in W^{1,\infty}([0, T]; H_0^1(\Omega))$ and $u_n' \in W^{1,\infty}([0, T]; L^2(\Omega))$, then the uniform convergence in (2.50) implies that $u \in C([0, T]; H_0^1(\Omega))$ and $u' \in C([0, T]; L^2(\Omega))$. In addition, (2.50) shows $u_n(x, 0) \to u(x, 0)$ in $H_0^1(\Omega)$, and since $u_0^n(x, 0) \to u_0(x, 0)$ in $H_0^1(\Omega)$, then $u(x, 0) = u_0(x, 0) \in H^1(\Omega)$. Also, since $u(x, t) = u_0(x, t)$ for $t < 0$, it follows that $u(x, t) = u_0(x, t)$ for all $t \leq 0$. This completes the proof of the local existence statement in Theorem 1.3. $\square$

Remark 2.6. It can be easily shown, the weak solutions obtained in the proof above, satisfy the following energy **inequality**:

$$E(t) + \int_0^t \int_\Omega g(u_t)u_t dxd\tau \leq E(0) + \int_0^t \int_\Omega f(u)u_t dxd\tau. \tag{2.51}$$

In the next section, we shall prove all weak solutions of (1.1) in the sense of Definition 1.2 satisfy the **energy identity** (1.6) and we will use this fact to justify the uniqueness of solutions.

3. Uniqueness of weak solutions

3.1. Energy identity

In order to prove the uniqueness of weak solutions, we shall justify the energy identity (1.6) rigorously. Notice that the energy identity can be derived formally by testing Eq. (1.1) by u_t, however, such calculation is not rigorous, due to the fact that u_t is not sufficiently regular to be the test function in as required in Definition 1.2. To resolve this issue, we employ the operator $T_\epsilon := (I - \epsilon\Delta)^{-1}$ to smooth functions in space. Some important properties of T_ϵ can be found in Appendix A. Denote $u_\epsilon = T_\epsilon u$.

We prove the energy identity (1.6) as follows.

Proof. Act the regularizing operator T_ϵ on every term of the equation and multiply by u'_ϵ. After integrating in space and time, we obtain

$$\int_0^t \int_\Omega u''_\epsilon(\tau) \cdot u'_\epsilon(\tau)dxd\tau + \int_0^t \int_\Omega \nabla u_\epsilon(\tau) \cdot \nabla u'_\epsilon(\tau)dxd\tau$$

$$+ \int_0^t \int_0^\infty \int_\Omega \nabla\big(u_\epsilon(\tau) - u_\epsilon(\tau - s)\big) \cdot \nabla u'_\epsilon(\tau)dx\mu(s)dsd\tau + \int_0^t \int_\Omega T_\epsilon\big(g\big(u'(\tau)\big)\big)u'_\epsilon(\tau)dxd\tau$$

$$= \int_0^t \int_\Omega T_\epsilon\big(f\big(u(\tau)\big)\big)u'_\epsilon(\tau)dxd\tau. \tag{3.1}$$

Since $u \in H_0^1(\Omega)$ and $u' \in L^2(\Omega)$, by means of Proposition A.1, we have $u_\epsilon \to u$ in $H_0^1(\Omega)$ and $u'_\epsilon \to u'$ in $L^2(\Omega)$. Thus, the first two integrals on the left-hand side of (3.1) converge to

$$\frac{1}{2}\big(\big\|u'(t)\big\|_2^2 - \big\|u'(0)\big\|_2^2 + \big\|\nabla u(t)\big\|_2^2 - \big\|\nabla u(0)\big\|_2^2\big).$$

Recall $u' \in L^{m+1}(\Omega)$, $g(u') \in L^{\frac{m+1}{m}}(\Omega)$, $m \geq 1$. By Proposition A.2, one has $\|u'_\epsilon\|_{m+1} \leq \|u'\|_{m+1}$ and $u'_\epsilon \to u'$ in $L^{m+1}(\Omega)$, as well as $\|T_\epsilon(g(u'))\|_{\frac{m+1}{m}} \leq \|g(u')\|_{\frac{m+1}{m}}$ and $T_\epsilon(g(u')) \to g(u')$ in $L^{\frac{m+1}{m}}(\Omega)$. Consequently, we can pass to the limit for the damping term in (3.1) by using the Lebesgue Dominated Convergence Theorem. This shows

$$\lim_{\epsilon\to 0} \int_0^t \int_\Omega T_\epsilon\big(g\big(u'(\tau)\big)\big)u'_\epsilon(\tau)dxd\tau = \int_0^t \int_\Omega g\big(u'(\tau)\big)u'(\tau)dxd\tau.$$

On the other hand, $u \in H^1(\Omega)$ implies $f(u) \in L^{6/p}(\Omega)$ due to the growth rate of f is p. With the assumption $p\frac{m+1}{m} < 6$, we have $f(u) \in L^{\frac{m+1}{m}}(\Omega)$, and therefore similar as the damping term,

we can prove the convergence of the source term in (3.1). That is

$$\lim_{\epsilon \to 0} \int_0^t \int_\Omega T_\epsilon \big(f\big(u(\tau)\big)\big) u_\epsilon'(\tau) dx d\tau = \int_0^t \int_\Omega f\big(u(\tau)\big) u'(\tau) dx d\tau.$$

It remains to treat the memory term in (3.1). We split it as

$$\int_0^t \int_0^\infty \int_\Omega \nabla\big(u_\epsilon(\tau) - u_\epsilon(\tau - s)\big) \cdot \nabla u_\epsilon'(\tau) dx \mu(s) ds d\tau$$

$$= \int_0^t \int_0^\infty \int_\Omega \nabla\big(u_\epsilon(\tau) - u_\epsilon(\tau - s)\big) \cdot \nabla\big(u_\epsilon'(\tau) - u_\epsilon'(\tau - s)\big) dx \mu(s) ds d\tau$$

$$+ \int_0^t \int_0^\infty \int_\Omega \nabla\big(u_\epsilon(\tau) - u_\epsilon(\tau - s)\big) \cdot \nabla u_\epsilon'(\tau - s) dx \mu(s) ds d\tau. \tag{3.2}$$

Notice, by virtue of Proposition A.1, the first integral on the right-hand side of (3.2) converges to

$$\frac{1}{2} \int_0^\infty \big(\big\| \nabla\big(u(t) - u(t - s)\big) \big\|_2^2 - \big\| \nabla\big(u(0) - u(-s)\big) \big\|_2^2 \big) \mu(s) ds,$$

where we have used the assumption $u_0(x, t) \in L_\mu^2(\mathbb{R}^-, H_0^1(\Omega))$. Next, we consider the second term on the right-hand side of (3.2):

$$\int_0^t \int_0^\infty \int_\Omega \nabla\big(u_\epsilon(\tau) - u_\epsilon(\tau - s)\big) \cdot \nabla \partial_\tau u_\epsilon(\tau - s) dx \mu(s) ds d\tau$$

$$= \int_0^t \int_0^\infty \int_\Omega \nabla\big(u_\epsilon(\tau) - u_\epsilon(\tau - s)\big) \cdot \nabla \partial_s \big(u_\epsilon(\tau) - u_\epsilon(\tau - s)\big) dx \mu(s) ds d\tau$$

$$= \frac{1}{2} \int_0^t \int_0^\infty \big\| \nabla\big(u_\epsilon(\tau) - u_\epsilon(\tau - s)\big) \big\|_2^2 \big(-\mu'(s)\big) ds d\tau, \tag{3.3}$$

where we performed the integration by parts with respect to the time variable s and used the assumption $\mu(\infty) = 0$. Notice that (3.3) is non-negative due to $\mu'(s) \leq 0$. In order to see that the limit of (3.3) exists as $\epsilon \to 0$, we employ (3.1)–(3.3) to get:

$$0 \le \lim_{\epsilon \to 0} \frac{1}{2} \int\limits_0^t \int\limits_0^\infty \big\| \nabla \big(u_\epsilon(\tau) - u_\epsilon(\tau - s) \big) \big\|_2^2 \big(-\mu'(s) \big) ds d\tau$$

$$= -\lim_{\epsilon \to 0} \int\limits_0^t \int\limits_0^\infty \int\limits_\Omega \nabla \big(u_\epsilon(\tau) - u_\epsilon(\tau - s) \big) \cdot \nabla \big(u'_\epsilon(\tau) - u'_\epsilon(\tau - s) \big) dx \mu(s) ds d\tau$$

$$- \lim_{\epsilon \to 0} \int\limits_0^t \int\limits_\Omega u''_\epsilon(\tau) \cdot u'_\epsilon(\tau) dx d\tau - \lim_{\epsilon \to 0} \int\limits_0^t \int\limits_\Omega \nabla u_\epsilon(\tau) \cdot \nabla u'_\epsilon(\tau) dx d\tau$$

$$- \lim_{\epsilon \to 0} \int\limits_0^t \int\limits_\Omega T_\epsilon \big(g \big(u'(\tau) \big) \big) u'_\epsilon(\tau) dx d\tau + \lim_{\epsilon \to 0} \int\limits_0^t \int\limits_\Omega T_\epsilon \big(f \big(u(\tau) \big) \big) u'_\epsilon(\tau) dx d\tau < \infty,$$

$$(3.4)$$

since we have shown that each limit on the right-hand side of (3.4) converges to a finite value.

Applying the convergence of u_ϵ in $H_0^1(\Omega)$ and (3.4), it follows from Fatou's Lemma that

$$\frac{1}{2} \int\limits_0^t \int\limits_0^\infty \big\| \nabla \big(u(\tau) - u(\tau - s) \big) \big\|_2^2 \big(-\mu'(s) \big) ds d\tau$$

$$\le \lim_{\epsilon \to 0} \frac{1}{2} \int\limits_0^t \int\limits_0^\infty \big\| \nabla \big(u_\epsilon(\tau) - u_\epsilon(\tau - s) \big) \big\|_2^2 \big(-\mu'(s) \big) ds d\tau$$

$$\le \frac{1}{2} \int\limits_0^t \int\limits_0^\infty \big\| \nabla \big(u(\tau) - u(\tau - s) \big) \big\|_2^2 \big(-\mu'(s) \big) ds d\tau, \qquad (3.5)$$

where the last inequality is due to the fact $\| \nabla (u_\epsilon(\tau) - u_\epsilon(\tau - s)) \|_2 \le \| \nabla (u(\tau) - u(\tau - s)) \|_2$ by means of Proposition A.1. This implies that the inequalities in (3.5) are actually equalities, which gives us the desired limit of (3.3) as $\epsilon \to 0$. $\quad \square$

3.2. Continuous dependence on initial data

This subsection is devoted to the proof of Theorem 1.4, which states that the solution of system (1.1) depends continuously on the initial data. The uniqueness of solutions then follows immediately.

Proof. Let u_0^n, $u_0 \in L_\mu^2(\mathbb{R}^-, H_0^1(\Omega))$, $n \in \mathbb{N}$, such that $u_0^n \to u_0$ in $L_\mu^2(\mathbb{R}^-, H_0^1(\Omega))$ with $u_0^n(0) \to u_0(0)$ in $H_0^1(\Omega)$ and in $L^{\frac{3(p-1)}{2}}(\Omega)$, $\frac{d}{dt} u_0^n(0) \to \frac{d}{dt} u_0(0)$ in $L^2(\Omega)$. Let $\{u_n\}$ and u be weak solutions on $[0, T]$ corresponding to the initial data $\{u_0^n\}$ and u_0, respectively. It is important to note here that the local existence time T can be selected independent of n. To see this, recall (2.27) and (2.29), which imply that T depends on K, while K depends on the initial energy. Nonetheless, we can choose K sufficiently large so that $K^2 \ge 4(E_n(0) + 1)$, for all n,

and $K^2 \geq 4(E(0) + 1)$, where $E_n(t)$ and $E(t)$ are quadratic energies corresponding to u_n and u, respectively. Thus, K and T are both independent of n. Furthermore, (2.36) and (2.37) yield

$$\begin{cases} E_n(t) \leq K^2/2, \quad E(t) \leq K^2/2, \quad \text{for all } t \in [0, T]; \\ \int_0^T \|u_n'\|_{m+1}^{m+1} dt \leq C_K, \quad \int_0^T \|u'\|_{m+1}^{m+1} dt \leq C_K, \end{cases} \tag{3.6}$$

for all $n \in \mathbb{N}$.

Now put: $\tilde{u}_n = u_n - u$ and

$$\tilde{E}_n(t) := \frac{1}{2}\left(\|\tilde{u}_n'(t)\|_2^2 + \|\nabla \tilde{u}_n(t)\|_2^2 + \int_0^\infty \|\nabla \tilde{w}_n(t, s)\|_2^2 \mu(s) ds \right),$$

where $\tilde{w}_n(x, t, s) = \tilde{u}_n(x, t) - \tilde{u}_n(x, t - s)$. By assumption, $\tilde{E}_n(0) \to 0$. We aim to show $\tilde{E}_n(t) \to 0$ uniformly on $[0, T]$.

Following the same approach in proving the energy identity (1.6), one can obtain that:

$$\tilde{E}_n(t) + \int_0^t \int_\Omega (g(u_n') - g(u'))(u_n' - u') dx d\tau - \frac{1}{2}\int_0^t \int_0^\infty \|\nabla \tilde{w}_n\|_2^2 \mu'(s) ds d\tau$$

$$= \tilde{E}_n(0) + \int_0^t \int_\Omega (f(u_n) - f(u))(u_n' - u') dx d\tau.$$

By the monotonicity of the function g and the assumption that $\mu' < 0$, then the following energy inequality holds:

$$\tilde{E}_n(t) \leq \tilde{E}_n(0) + \int_0^t \int_\Omega (f(u_n) - f(u))\tilde{u}_n' dx d\tau. \tag{3.7}$$

If $1 \leq p \leq 3$, then the uniqueness of weak solutions can be obtained immediately by the energy inequality (3.7). To see this, we recall, if $1 \leq p \leq 3$, then $f : H_0^1(\Omega) \to L^2(\Omega)$ is locally Lipschitz continuous, and along with (3.6), we infer

$$\int_0^t \int_\Omega (f(u_n) - f(u))\tilde{u}_n' dx d\tau$$

$$\leq C(K)\left(\int_0^t \|\nabla \tilde{u}_n\|_2^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\tilde{u}_n'\|_2^2 d\tau \right)^{\frac{1}{2}} \leq C(K) \int_0^t \tilde{E}_n(\tau) d\tau.$$

Thus, it follows from (3.7) that

$$\tilde{E}_n(t) \leq \tilde{E}_n(0) + C(K) \int\limits_0^t \tilde{E}_n(\tau)d\tau, \quad \text{for all } t \in [0, T].$$

By Gronwall's inequality, we have

$$\tilde{E}(t) \leq C(K, T)\tilde{E}_n(0)$$

for all $t \in [0, T]$. Since $\tilde{E}_n(0) \to 0$, then $\tilde{E}_n(t) \to 0$ uniformly on $[0, T]$, for the case $1 \leq p \leq 3$.

However, if $3 < p < 6$, the estimate for the source term is more subtle. Here, we follow a clever idea that has been used in [7]. As in [7], we shall perform integration by parts *twice* with respect to the time variable t, which essentially convert $\tilde{u}_n' \in L^2(\Omega)$ to the more regular term $\tilde{u}_n \in H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. More precisely, we compute as follows:

$$\int\limits_0^t \int\limits_\Omega \big(f(u_n) - f(u)\big)\tilde{u}_n' dx d\tau$$

$$= \left[\int\limits_\Omega \big(f(u_n) - f(u)\big)\tilde{u}_n dx\right]_0^t - \int\limits_0^t \int\limits_\Omega \big(f_1'(u_n)u_n' - f'(u)u'\big)\tilde{u}_n dx d\tau$$

$$= \left[\int\limits_\Omega \big(f(u_n) - f(u)\big)\tilde{u}_n dx\right]_0^t - \int\limits_0^t \int\limits_\Omega \big(f'(u_n) - f'(u)\big)u_n'\tilde{u}_n dx d\tau$$

$$- \int\limits_0^t \int\limits_\Omega f'(u)\tilde{u}_n'\tilde{u}_n dx d\tau$$

$$= \left[\int\limits_\Omega \big(f(u_n) - f(u)\big)\tilde{u}_n dx\right]_0^t - \int\limits_0^t \int\limits_\Omega \big(f'(u_n) - f'(u)\big)u_n'\tilde{u}_n dx d\tau$$

$$- \frac{1}{2}\left[\int\limits_\Omega f'(u)|\tilde{u}_n|^2 dx\right]_0^t + \frac{1}{2}\int\limits_0^t \int\limits_\Omega f''(u)u'|\tilde{u}_n|^2 dx d\tau. \tag{3.8}$$

By the assumptions on f, we have

$$\begin{cases} \big|f''(u)\big| \leq C\big(|u|^{p-2} + 1\big) \\ \big|f'(u)\big| \leq C\big(|u|^{p-1} + 1\big) \\ \big|f(u_n) - f(u)\big| \leq C\big(|u_n|^{p-1} + |u|^{p-1} + 1\big)|\tilde{u}_n| \\ \big|f'(u_n) - f'(u)\big| \leq C\big(|u_n|^{p-2} + |u|^{p-2} + 1\big)|\tilde{u}_n|. \end{cases} \tag{3.9}$$

By using (3.9), then we estimate (3.8) as follows:

$$\int_0^t \int_\Omega \left(f(u_n) - f(u) \right) \tilde{u}_n' \, dx \, d\tau$$

$$\leq \int_\Omega \left(\left| \tilde{u}_n(t) \right|^2 + \left| \tilde{u}_n(0) \right|^2 \right) dx + \int_\Omega \left(\left| u_n(t) \right|^{p-1} + \left| u(t) \right|^{p-1} \right) \left| \tilde{u}_n(t) \right|^2 dx$$

$$+ \int_\Omega \left(\left| u_n(0) \right|^{p-1} + \left| u(0) \right|^{p-1} \right) \left| \tilde{u}_n(0) \right|^2 dx + \int_0^t \int_\Omega \left| \tilde{u}_n \right|^2 \left(\left| u_n' \right| + \left| u' \right| \right) dx \, d\tau$$

$$+ \int_0^t \int_\Omega \left(\left| u_n \right|^{p-2} + |u|^{p-2} \right) \left| \tilde{u}_n \right|^2 \left(\left| u_n' \right| + \left| u' \right| \right) dx \, d\tau$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.10}$$

The next step is to estimate each term on the right hand side of (3.10). First, let us look at

$$I_1 = \int_\Omega \left(\left| \tilde{u}_n(t) \right|^2 + \left| \tilde{u}_n(0) \right|^2 \right) dx = \int_\Omega \left(\left| \tilde{u}_n(0) + \int_0^t \tilde{u}_n'(\tau) d\tau \right|^2 + \left| \tilde{u}_n(0) \right|^2 \right) dx$$

$$\leq 3 \left\| \tilde{u}_n(0) \right\|_2^2 + 2t \int_0^t \left\| \tilde{u}_n'(\tau) \right\|_2^2 d\tau \leq C \left(\tilde{E}_n(0) + T \int_0^t \tilde{E}_n(\tau) d\tau \right). \tag{3.11}$$

Also, by Hölder's inequality and the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, one has

$$I_3 = \int_\Omega \left(\left| u_n(0) \right|^{p-1} + \left| u(0) \right|^{p-1} \right) \left| \tilde{u}_n(0) \right|^2 dx$$

$$\leq \left(\left\| u_n(0) \right\|_{\frac{3(p-1)}{2}}^{p-1} + \left\| u(0) \right\|_{\frac{3(p-1)}{2}}^{p-1} \right) \left\| \tilde{u}_n(0) \right\|_6^2 \leq C \tilde{E}_n(0), \tag{3.12}$$

where we have used the fact $u_n(0) = u_0^n(0) \to u(0) = u_0(0)$ in $L^{\frac{3(p-1)}{2}}(\Omega)$.

Similarly,

$$I_4 = \int_0^t \int_\Omega \left| \tilde{u}_n \right|^2 \left(\left| u_n' \right| + \left| u' \right| \right) dx \, d\tau \leq C \int_0^t \left\| \tilde{u}_n \right\|_6^2 \left(\left\| u_n' \right\|_2 + \left\| u' \right\|_2 \right) d\tau$$

$$\leq C \int_0^t \left\| \nabla \tilde{u}_n \right\|_2^2 \left(\left\| u_n' \right\|_2 + \left\| u' \right\|_2 \right) d\tau \leq C(K) \int_0^t \tilde{E}_n(\tau) d\tau. \tag{3.13}$$

To estimate I_5, we recall the assumption $p \frac{m+1}{m} < 6$, which implies $\frac{6}{6-p} < m + 1$. Hence,

$$I_5 = \int_0^t \int_\Omega \left(\left| u_n \right|^{p-2} + |u|^{p-2} \right) \left| \tilde{u}_n \right|^2 \left(\left| u_n' \right| + \left| u' \right| \right) dx \, d\tau$$

$$\leq C \int_0^t \left(\|u_n\|_6^{p-2} + \|u\|_6^{p-2} \right) \|\tilde{u}_n\|_6^2 \left(\|u_n'\|_{\frac{6}{6-p}} + \|u'\|_{\frac{6}{6-p}} \right) d\tau$$

$$\leq C(K) \int_0^t \tilde{E}_n(\tau) \left(\|u_n'\|_{m+1} + \|u'\|_{m+1} \right) d\tau. \tag{3.14}$$

Finally, we estimate $I_2 = \int_\Omega (|\tilde{u}_n(t)|^{p-1} + |u(t)|^{p-1}) |\tilde{u}_n(t)|^2 dx$. For the sake of clarification, we focus on the term $\int_\Omega |\tilde{u}_n(t)|^{p-1} |\tilde{u}_n(t)|^2 dx$. The estimate for $\int_\Omega |u(t)|^{p-1} |\tilde{u}_n(t)|^2 dx$ will be the same. There are two different cases to be considered.

Case 1. $3 < p < 5$. In this case, we split the integral to obtain

$$\int_\Omega |u_n(t)|^{p-1} |\tilde{u}_n(t)|^2 dx \leq \int_\Omega |\tilde{u}_n(t)|^2 dx + \int_{\{x \in \Omega : |\tilde{u}_n(t)| > 1\}} |u_n(t)|^{p-1} |\tilde{u}_n(t)|^2 dx. \tag{3.15}$$

Note that the first term on the right hand side of (3.15) has been estimated in (3.11). So, we only consider the second term. Let $\epsilon \in (0, 5-p)$, so if $|u_n| > 1$, then $|u_n|^{p-1} < |u_n|^{4-\epsilon}$. It follows that

$$\int_{\{x \in \Omega : |u(t)| > 1\}} |u_n(t)|^{p-1} |\tilde{u}_n(t)|^2 dx \leq \int_\Omega |u_n(t)|^{4-\epsilon} |\tilde{u}_n(t)|^2 dx \leq \|u_n(t)\|_6^{4-\epsilon} \|\tilde{u}_n(t)\|_{\frac{6}{1+\epsilon/2}}^2$$

$$\leq C \|\nabla u_n(t)\|_2^{4-\epsilon} \|\tilde{u}_n(t)\|_{H^{1-\epsilon/4}(\Omega)}^2$$

$$= C(K) \left(\epsilon \|\nabla \tilde{u}_n(t)\|_2^2 + C_\epsilon \|\tilde{u}_n(t)\|_2^2 \right) \tag{3.16}$$

where we have use the imbedding $H^{1-\delta}(\Omega) \hookrightarrow L^{\frac{6}{1+2\delta}}(\Omega)$ and the interpolation inequality. We infer from (3.11), (3.15) and (3.16) that

$$\int_\Omega |u_n(t)|^{p-1} |\tilde{u}_n(t)|^2 dx \leq C(K, \epsilon) \left(\tilde{E}_n(0) + T \int_0^t \tilde{E}_n(\tau) d\tau \right) + C(K) \epsilon \tilde{E}_n(t). \tag{3.17}$$

Case 2. $5 \leq p < 6$. In this case, we require the initial data $u_0^n(0), u_0(0) \in L^{\frac{3(p-1)}{2}}(\Omega)$. Note, for any $\epsilon > 0$, there exists $\phi \in C_0(\Omega)$ such that $\|u_0(0) - \phi\|_{\frac{3(p-1)}{2}} < \epsilon^{\frac{1}{p-1}}$. We consider

$$\int_\Omega |u_n(t)|^{p-1} |\tilde{u}_n(t)|^2 dx$$

$$\leq C \int_\Omega |u_n(t) - u_0^n(0)|^{p-1} |\tilde{u}_n(t)|^2 dx + C \int_\Omega |u_0^n(0) - u_0(0)|^{p-1} |\tilde{u}_n(t)|^2 dx$$

$$+ C \int_\Omega |u_0(0) - \phi|^{p-1} |\tilde{u}_n(t)|^2 dx + C \int_\Omega |\phi|^{p-1} |\tilde{u}_n(t)|^2 dx. \tag{3.18}$$

By the assumption $p\frac{m+1}{m} < 6$ and $5 \le p < 6$, we infer $m > 5$. In addition, we have

$$\frac{3(p-1)}{2(m+1)} < \frac{3(5m-1)}{2(m+1)^2} < 1,$$

for $m > 5$. Therefore, the first term on the right hand side of (3.18) can be estimated as follows:

$$\int_\Omega |u_n(t) - u_0^n(0)|^{p-1} |\tilde{u}_n(t)|^2 \le \left(\int_\Omega |u_n(t) - u_n(0)|^{\frac{3(p-1)}{2}} \right)^{2/3} \|\tilde{u}_n(t)\|_6^2$$

$$\le C \left(\int_\Omega \left| \int_0^t u_n'(\tau) d\tau \right|^{\frac{3(p-1)}{2}} dx \right)^{2/3} \|\tilde{u}_n(t)\|_{1,\Omega}^2$$

$$\le C \left[\int_\Omega \left(\int_0^t |u_n'|^{m+1} d\tau \right)^{\frac{3(p-1)}{2(m+1)}} dx \right]^{2/3} T^{\frac{m(p-1)}{m+1}} \tilde{E}_n(t)$$

$$\le C(K) T^{\frac{m(p-1)}{m+1}} \tilde{E}_n(t), \qquad (3.19)$$

where we have used the bound $\int_0^T \|u_n'\|_{m+1}^{m+1} dt \le K$ and the fact that $\frac{3(p-1)}{2(m+1)} < 1$.

Next, we consider the second term on the right hand side of (3.18).

$$\int_\Omega |u_0^n(0) - u_0(0)|^{p-1} |\tilde{u}_n(t)|^2 dx \le \|u_0^n(0) - u_0(0)\|_{\frac{3(p-1)}{2}}^{p-1} \|\tilde{u}_n(t)\|_6^2 \le \epsilon \tilde{E}_n(t), \quad (3.20)$$

for n sufficiently large, due to the assumption $u_0^n(0) \to u_0(0)$ in $L^{\frac{3(p-1)}{2}}(\Omega)$.

Similarly, we have

$$\int_\Omega |u_0(0) - \phi|^{p-1} |\tilde{u}_n(t)|^2 dx \le \|u_0(0) - \phi\|_{\frac{3(p-1)}{2}}^{p-1} \|\tilde{u}_n(t)\|_6^2 \le C\epsilon \tilde{E}_n(t). \qquad (3.21)$$

In addition, since $\phi \in C_0(\Omega)$, it is clear that $|\phi(x)| \le C(\epsilon)$, for all $x \in \Omega$. So, by (3.11), we estimate the last term on the right hand side of (3.18) as follows:

$$\int_\Omega |\phi|^{p-1} |\tilde{u}_n(t)|^2 dx \le C(\epsilon) \int_\Omega |\tilde{u}_n(t)|^2 dx \le C(\epsilon) \left(\tilde{E}_n(0) + T \int_0^t \tilde{E}_n(\tau) d\tau \right). \qquad (3.22)$$

It follows from (3.18)–(3.22)) that

$$\int_\Omega |\tilde{u}_n(t)|^{p-1} |\tilde{u}_n(t)|^2 dx$$

$$\le C(K)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon \right) \tilde{E}_n(t) + C(\epsilon) \left(\tilde{E}_n(0) + T \int_0^t \tilde{E}_n(\tau) d\tau \right), \qquad (3.23)$$

in the case $5 \le p < 6$.

By combining (3.17) and (3.23) for the both cases, we conclude that,

$$I_2 \leq C(K)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{E}_n(t) + C(K, \epsilon)\left(\tilde{E}_n(0) + T \int\limits_0^t \tilde{E}_n(\tau)d\tau\right), \qquad (3.24)$$

for any $3 < p < 6$.

Now, by combining (3.7), (3.10)–(3.14) and (3.24), we have

$$\tilde{E}_n(t) \leq C(K)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{E}_n(t) + C(K, \epsilon)\tilde{E}_n(0)$$

$$+ C(K, T, \epsilon) \int\limits_0^t \tilde{E}_n(\tau)\left(\|u_n'\|_{m+1} + \|u'\|_{m+1} + 1\right)d\tau,$$

for all $t \in [0, T]$. By selecting ϵ and T sufficiently small so that

$$C(K)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right) < 1,$$

then by Gronwall's inequality,

$$\tilde{E}_n(t) \leq C(K, T, \epsilon)\tilde{E}_n(0) \exp\left(\int\limits_0^t \left(\|u_n'\|_{m+1} + \|u'\|_{m+1} + 1\right)d\tau\right).$$

Hence,

$$\tilde{E}_n(t) \leq C(K, T, \epsilon)\tilde{E}_n(0),$$

and since $\tilde{E}_n(0) \to 0$, we conclude that $\tilde{E}_n(t) \to 0$ uniformly on $[0, T]$.   □

4. Global existence

In this section we prove Theorem 1.6 stating that a local weak solution u on $[0, T]$ can be extended to $[0, \infty)$ provided the damping term $g(u_t)$ dominates the source $f(u)$, i.e., $m \geq p$.

Proof. One may employ a standard continuation argument from ODE theory to obtain that the weak solution u is either global or there exists $0 < T_{max} < \infty$ such that

$$\limsup_{t \to T_{max}^-} \mathcal{E}(t) = +\infty \qquad (4.1)$$

where $\mathcal{E}(t)$ is the modified energy defined by

$$\mathcal{E}(t) = E(t) + \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} \qquad (4.2)$$

and the quadratic energy $E(t)$ is as defined in (1.7). We aim to show the latter cannot happen if $m \geq p$.

Let u be a weak solution to (1.1) on $[0, T]$ in the sense of Definition 1.2. With the modified energy $\mathcal{E}(t)$, the energy identity (1.6) now reads

$$\mathcal{E}(t) + \int_0^t \int_\Omega g(u_t)u_t dx d\tau - \frac{1}{2} \int_0^t \int_0^\infty \|\nabla w\|_2^2 \mu'(s) ds d\tau$$

$$= \mathcal{E}(0) + \int_0^t \int_\Omega f(u)u_t dx d\tau + \int_0^t \int_\Omega |u|^{p-1} u u_t dx d\tau. \tag{4.3}$$

By the assumption $|f(u)| \leq C(|u|^p + 1)$, we deduce

$$\left| \int_0^t \int_\Omega f(u)u_t dx d\tau \right| \leq C \int_0^t \int_\Omega (|u|^p + 1)|u_t| dx d\tau$$

$$\leq \int_0^t \|u_t\|_{p+1} \left(\|u\|_{p+1}^p + |\Omega|^{\frac{p}{p+1}} \right) d\tau$$

$$\leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t \left(\|u\|_{p+1}^{p+1} + |\Omega| \right) d\tau$$

$$\leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t \mathcal{E}(\tau) d\tau + C_\epsilon T |\Omega|, \tag{4.4}$$

where $\epsilon > 0$ to be chosen later. Similarly, we have

$$\left| \int_0^t \int_\Omega |u|^{p-1} u u_t dx d\tau \right| \leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t \mathcal{E}(\tau) d\tau. \tag{4.5}$$

It follows from (4.4)–(4.5) and the assumptions $g(s)s \geq a|s|^{m+1}$, $\mu(s) \leq 0$, we infer from (4.3) that

$$\mathcal{E}(t) + a \int_0^t \|u_t\|_{m+1}^{m+1} d\tau \leq \mathcal{E}(0) + \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t \mathcal{E}(\tau) d\tau + C_{T,\epsilon}$$

$$\leq \mathcal{E}(0) + \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \mathcal{E}(\tau) d\tau + C_{T,\epsilon}, \tag{4.6}$$

where we have used the assumption $m \geq p$, Hölder's and Young's inequalities.

Now, if we choose $\epsilon < a$, then (4.6) yields

$$\mathcal{E}(t) \leq \mathcal{E}(0) + C_\epsilon \int_0^t \mathcal{E}(\tau) d\tau + C_{T,\epsilon}.$$

By Gronwall's inequality, we conclude

$$\mathcal{E}(t) \leq \big(\mathcal{E}(0) + C_{T,\epsilon}\big)e^{C_\epsilon T},$$

for all $t \in [0, T]$. Thus, (4.1) cannot happen, which implies that u is a global weak solution. □

Acknowledgments

Appendix A

Here we provide some properties of the regularizing operator $(I - \epsilon\Delta)^{-1}$ that we used in the justification of the energy identity. Let $u \in L^p(\Omega)$, $1 < p < \infty$. Set $u_\epsilon := (I - \epsilon\Delta)^{-1}u$ with $u_\epsilon = 0$ on $\partial\Omega$.

Proposition A.1. *The following statements hold.*

- *If $u \in L^2(\Omega)$, then $\|u_\epsilon\|_2 \leq \|u\|_2$, and $u_\epsilon \to u$ in $L^2(\Omega)$, as $\epsilon \to 0$.*
- *If $u \in H_0^1(\Omega)$, then $\|u_\epsilon\|_{H_0^1} \leq \|u\|_{H_0^1}$, and $u_\epsilon \to u$ in $H_0^1(\Omega)$, as $\epsilon \to 0$.*

Proof. The proof of these two statements are essentially the same. We only consider the second one. Since Ω is bounded with boundary of class C^2, it is well known that $-\Delta$, with the domain $H^2(\Omega) \cap H_0^1(\Omega)$, is positive, self-adjoint, and it is the inverse of a compact operator. Moreover, $-\Delta$ has an infinite sequence of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$, and a corresponding sequence of eigenfunctions $\{e_j : j = 1, 2, \cdots\}$ that forms an orthonormal basis for $L^2(\Omega)$, i.e., $-\Delta e_j = \lambda_j e_j$ with $e_j = 0$ on $\partial\Omega$. Also, the sequence $\{e_j : j = 1, 2, \cdots\}$ is an orthogonal basis for $H_0^1(\Omega)$. In addition, the standard norm $\|u\|_{H_0^1(\Omega)}$ is equivalent to $(\sum_{j=1}^\infty \lambda_j |(u, e_j)|^2)^{1/2}$. Therefore, we consider

$$\|u\|_{H_0^1(\Omega)}^2 = \sum_{j=1}^\infty \lambda_j \big|(u, e_j)\big|^2.$$

We now have

$$\|u_\epsilon\|^2_{H^1_0(\Omega)} = \sum_{j=1}^{\infty} \frac{\lambda_j}{(1+\epsilon\lambda_j)^2}\big|(u,e_j)\big|^2 \le \sum_{j=1}^{\infty} \lambda_j \big|(u,e_j)\big|^2 = \|u\|^2_{H^1_0}.$$

To see that $u_\epsilon \to u$ in $H^1_0(\Omega)$, we calculate

$$\begin{aligned}
\|u_\epsilon - u\|^2_{H^1_0(\Omega)} &= \sum_{j=1}^{\infty}\left(\frac{\epsilon\lambda_j}{1+\epsilon\lambda_j}\right)^2 \lambda_j\big|(u,e_j)\big|^2 \\
&= \sum_{j=1}^{N}\left(\frac{\epsilon\lambda_j}{1+\epsilon\lambda_j}\right)^2 \lambda_j\big|(u,e_j)\big|^2 + \sum_{j=N+1}^{\infty}\left(\frac{\epsilon\lambda_j}{1+\epsilon\lambda_j}\right)^2 \lambda_j\big|(u,e_j)\big|^2 \\
&\le \epsilon^2 \sum_{j=1}^{N}\lambda_j^3\big|(u,e_j)\big|^2 + \sum_{j=N+1}^{\infty}\lambda_j\big|(u,e_j)\big|^2.
\end{aligned}$$

Since $u \in H^1_0(\Omega)$, then $\sum_{j=1}^{\infty} \lambda_j|(u,e_j)|^2 < \infty$. Thus, by choosing N large enough and then selecting ϵ small enough, the conclusion follows. $\square$

Proposition A.2. *Let $u \in L^p(\Omega)$ with $1 < p < \infty$, then $\|u_\epsilon\|_p \le \|u\|_p$ and $u_\epsilon \to u$ in $L^p(\Omega)$, as $\epsilon \to 0$.*

Proof. By the definition of u_ϵ, we have

$$\begin{cases} u_\epsilon - \epsilon\Delta u_\epsilon = u & \text{in } \Omega; \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

From the standard elliptic theory (see [22]), $u_\epsilon \in W^{2,p}(\Omega)$ whenever $u \in L^p(\Omega)$, for $1 < p < \infty$.

Case 1: $2 \le p < \infty$.
Multiply the equation by $|u_\epsilon|^{p-2}u_\epsilon$ and integration by parts:

$$\|u_\epsilon\|^p_p + \epsilon\int_\Omega \nabla u_\epsilon \cdot \nabla\big(|u_\epsilon|^{p-2}u_\epsilon\big)dx = \int_\Omega u|u_\epsilon|^{p-2}u_\epsilon dx.$$

A straightforward calculation gives

$$\|u_\epsilon\|^p_p + \epsilon(p-1)\int_\Omega |u_\epsilon|^{p-2}|\nabla u_\epsilon|^2 dx = \int_\Omega u|u_\epsilon|^{p-2}u_\epsilon dx.$$

Since the middle term is positive, one has

$$\|u_\epsilon\|^p_p \le \int_\Omega |u||u_\epsilon|^{p-1}dx \le \|u\|_p\|u_\epsilon\|^{p-1}_p.$$

It follows that

$$\|u_\epsilon\|_p \leq \|u\|_p, \quad \text{for } 2 \leq p < \infty. \tag{A.1}$$

To show $u_\epsilon \to u$ in L^p, we argue by contradiction. Suppose not, then there exist $\delta > 0$ and a subsequence $\{u_{\epsilon_j}\}$ such that

$$\|u_{\epsilon_j} - u\|_p \geq \delta, \quad \text{for all } j = 1, 2, \cdots. \tag{A.2}$$

Due to the uniform bound (A.1) and the fact $u_\epsilon \to u$ strongly in $L^2(\Omega)$ from Proposition A.1, we can extract a further subsequence $\{u_{\epsilon_{j_k}}\}$ such that $u_{\epsilon_{j_k}} \to u$ weakly in $L^p(\Omega)$. As a result, the weak convergence and the uniform bound (A.1) imply

$$\|u\|_p \leq \liminf_{k \to 0} \|u_{\epsilon_{j_k}}\|_p \leq \limsup_{k \to 0} \|u_{\epsilon_{j_k}}\|_p \leq \|u\|_p.$$

This shows

$$\lim_{k \to 0} \|u_{\epsilon_{j_k}}\|_p = \|u\|_p.$$

Since we already know $u_{\epsilon_{j_k}} \to u$ weakly in $L^p(\Omega)$, it follows that $u_{\epsilon_{j_k}} \to u$ strongly in $L^p(\Omega)$, which violates (A.2).

Case 2: $1 < p < 2$.
In this case, the conjugate $2 < p^* < \infty$. We calculate

$$\|u_\epsilon\|_p = \sup_{\|\varphi\|_{p^*}=1} \int_\Omega u_\epsilon \varphi \, dx = \sup_{\|\varphi\|_{p^*}=1} \int_\Omega u_\epsilon \big((I - \epsilon\Delta)\varphi_\epsilon\big) dx = \sup_{\|\varphi\|_{p^*}=1} \int_\Omega \big((I - \epsilon\Delta)u_\epsilon\big)\varphi_\epsilon \, dx$$

$$= \sup_{\|\varphi\|_{p^*}=1} \int_\Omega u\varphi_\epsilon \, dx \leq \sup_{\|\varphi\|_{p^*}=1} \|u\|_p \|\varphi_\epsilon\|_{p^*} \leq \sup_{\|\varphi\|_{p^*}=1} \|u\|_p \|\varphi\|_{p^*} = \|u\|_p,$$

where the uniform bound (A.1) has been used.

Finally, arguing as in Case 1, the convergence of u_ϵ in $L^p(\Omega)$ for $1 < p < 2$ follows. □

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