Exam I

Differential Equations, MAP 2302

Sep 15, 2016

Name: Key ID:

- You have 75 minutes to complete the exam.
- The exam is closed books and notes. You must show supporting work to get full and partial credits.

Problem 1. (20 points) Solve the following differential equation

$$(6xy + y^2 - \sin x)dx + (3x^2 + 2xy - y^2)dy = 0.$$

$M_y = 6x + 2y = N_x \Rightarrow$ exact. 5 (as long as they say "exact")

$F = \int 6xy + y^2 - \sin x \, dx = 3x^2y - xy^2 + \cos x + f(y)$ 5

$F = \int 3x^2 + 2xy - y^2 dy = 3x^2y + xy^2 - \frac{1}{3}y^3 + g(x)$ 5

Sol

$3x^2y + xy^2 - \frac{1}{3}y^3 + \cos x = C$ 5

Problem 2. (20 points) Solve the following differential equation with initial value

$$x\frac{dy}{dx} + (2x^2+1)y = 2x, \quad y(1) = 1/2.$$

~~Homogeneous: Since~~

linear since $\frac{dy}{dx} + \frac{2x^2+1}{x} y = 2$ 3)

w/ $P(x) = 2x + \frac{1}{x}$ $Q(x) = 2$.

$\mu = e^{\int p(x)dx} = e^{\int 2x + \frac{1}{x} dx} = e^{x^2 + \ln x}$ 5)

$= e^{x^2} \cdot x$

(you may assume $x>0$ due to initial value)

$\frac{d}{dx}(e^{x^2} x\, y) = 2xe^{x^2}$ 5)

$e^{x^2} x\, y = \int 2xe^{x^2} dx = e^{x^2} + c$

IC: $e \cdot \frac{1}{2} = e + c \Rightarrow c = -\frac{1}{2}e$ 5)

$y = \frac{e^{x^2} - \frac{1}{2}e}{e^{x^2} x}$ 2)

Problem 3. (20 points) Solve the following differential equation

$$\left(x\tan\left(\frac{y}{x}\right)+y\right)dx - x\,dy = 0.$$

Homogeneous since

$$\frac{dy}{dx} = \tan\left(\frac{y}{x}\right) + \frac{y}{x}. \qquad 5$$

$y = vx$, $dy = v\,dx + x\,dv$

$(\tan v + v)\,dx - (v\,dx + x\,dv) = 0 \qquad 6$

$\tan v\,dx = x\,dv \longrightarrow$ separable

$\frac{1}{x}\,dx = \frac{1}{\tan v}\,dv = \frac{\cos v}{\sin v}\,dv$

$\ln|x| = +\ln|\sin v| + C \qquad 6$

$\ln|x| = \ln\left|\sin\left(\frac{y}{x}\right)\right| + C \qquad 3$

Problem 4. (20 points) Solve the following differential equation

$$(5x + 2y + 1)dx + (2x + y + 1)dy = 0.$$

2 $\frac{\mathrm{X}}{5} \neq \frac{1}{2} \Rightarrow \frac{a_2}{a_1} \neq \frac{b_2}{b_1}$

So let $x = X + h$, $y = Y + k$ w/ $\begin{cases} 5h + 2k + 1 = 0 \\ 2h + k + 1 = 0 \end{cases} \Rightarrow \begin{cases} h = 1 \\ k = -3 \end{cases}$

5 $(5X + 2Y)dX + (2X + Y)dY = 0 \rightarrow$ homogeneous

$Y = vX, \quad dY = v\,dX + X\,dv$

5 $[(5+2v)X + (2+v)Xv]\,dX + (2+v)X^2\,dv = 0$

$(5 + 4v + v^2)\,dX + (2+v)X\,dv = 0 \rightarrow$ separable

6 $-\frac{1}{X}dX = \frac{2+v}{5+4v+v^2}dv$

$-\ln|X| = \frac{1}{2}\ln|5+4v+v^2| + C$

2 $-\ln|x-1| = \frac{1}{2}\ln\left|5 + 4\cdot\frac{y+3}{x-1} + \left(\frac{y+3}{x-1}\right)^2\right| + C$

OR $\underbrace{M_y = N_x = 2}_{5}$ $\quad \underbrace{F = \frac{5}{2}x^2 + 2xy + x + \frac{1}{2}y^2 + y = C}_{15}$

Problem 5. (20 points) Solve the following differential equation

$$(5xy + 4y^2 + 1)dx + (x^2 + 2xy)dy = 0.$$

(Hint: find a special integrating factor $\mu(x)$.)

5

$$\frac{M_y - N_x}{N} = \frac{5x + 8y - 2x - 2y}{x^2 + 2xy} = \frac{3x + 6y}{x(x+2y)} = \frac{3(x+2y)}{x(x+2y)} = \frac{3}{x}$$

is of x only

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int \frac{3}{x} dx} = x^3.$$

5

$$\underline{x^3(5xy + 4y^2 + 1)}\,dx + \underline{x^3(x^2 + 2xy)}\,dy = 0. \rightarrow \text{exact.}$$

8

$$F = \int 5x^4y + 4x^3y^2 + 1\,dx = x^5y + x^4y^2 + x + f(y)$$

$$F = \int x^5 + 2x^4y\,dy = x^5y + x^4y^2 + g(x)$$

2

<u>Sol</u>: $x^5y + x^4y^2 + x = C$